



mathematical models and methods

unit 9

Simultaneous linear algebraic equations



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Unit 9

Simultaneous linear algebraic equations

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Introduction

The purpose of this unit is to examine one of the principal methods for solving sets of equations such as

$$1.2x_1 + 0.3x_2 - 1.3x_3 = 12 \quad E_1$$

$$3.3x_1 - 1.1x_2 + 2.4x_3 = 5.3 \quad E_2$$

$$4.2x_1 + 2.3x_2 + 7.3x_3 = 7.4 \quad E_3$$

These are called **simultaneous algebraic linear equations**. In this instance we have three equations, where the problem is to determine values for the unknown variables x_1 , x_2 and x_3 so that all three equations are satisfied *simultaneously*. They are called *algebraic* as their solutions are numbers (as compared with the sequences which were the solutions to the recurrence relations in Unit 1, or the functions which were the solutions to the differential equations in Units 2 and 6). Finally, the term *linear* derives from the case of two simultaneous equations in two unknowns, where each equation can be represented by a straight line.

For brevity, these sets of equations will just be referred to as **simultaneous equations**, and the individual equations as E_1 , E_2 , etc.

Equations of this form occur frequently in practice. The unknowns could be currents in an electrical network, for instance, or stresses in an engineering structure, or quantities of different goods transported from place to place. In such applications, we might be dealing with hundreds, or even thousands, of equations, in which case a computer is essential.

Simultaneous linear equations also occur frequently as steps in the solution of other kinds of mathematical problem: for example, selecting a particular solution of a recurrence relation or a differential equation by using initial conditions.

The discussion in this unit is largely confined to two equations in two unknowns, or three equations in three unknowns. However, we shall see that the method of solution chosen can easily be extended to larger sets of equations. There are many methods for solving simultaneous equations. We have chosen a particular version of the method called **Gaussian elimination**, for the following reasons:

- (i) because of its simplicity,
- (ii) because it is systematic and can easily be carried out on a computer (which is essential if it is to be of any practical use),
- (iii) because it clarifies the different types of solution that arise, and the various things that can go wrong.

Study guide

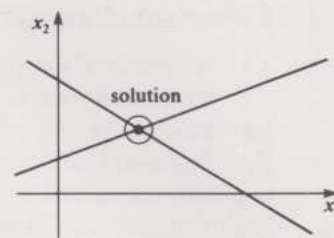
You should study Sections 1 and 2 and Subsection 3.1 before watching the television programme.

There are several ways of setting out Gaussian elimination, so even if you have met the method before, please read Section 1 to ensure that you can follow the way it is done in this unit.

Section 4 describes a computer package which will enable you to solve systems of linear equations by Gaussian elimination. One of the assignment questions may require you to use this package. We suggest that you combine this terminal visit with the one for Unit 10. Before making this visit, you should have studied Sections 1 to 4 of Unit 9 and Sections 1 and 2 of Unit 10.

The exercises at the end of each section are designed to give extra practice. It is not necessary to do all of them when you study the unit; indeed, you may wish to save some of them for revision purposes. However, you are encouraged to do the end of unit test immediately after studying the unit.

The five sections of this unit are not of equal length. You will probably find that Sections 1 and 5 each take you only about half as long as the other sections.



Units 1, 6



Carl Friedrich Gauss (1777–1855) was described by Laplace as 'the greatest mathematician in the world'. He made a large number of major contributions to mathematics and physical science. Amongst other things he devised the method of successive elimination for solving sets of simultaneous equations, a version of which is given in this unit.

Acknowledgement

Thanks are due to the National Physical Laboratory for several useful discussions in the course of preparing this unit.

1 Linear simultaneous algebraic equations and their solution

1.1 A ‘practical’ example

John, who is slimming, is planning a simple breakfast of cereal and sugar substitute. John is an eccentric character who drinks water with his breakfast, and eats his cereal dry. He has decided to allow himself exactly 500 calories and 10 grams of protein. The calorie and protein content of the foods are shown in the following table.

	Cereal	Sugar substitute
Calories per gram	6	2
Grams of protein per gram of food	0.1	0.1

He would like to know how many grams of cereal and of sugar substitute he should eat to satisfy his requirements.

Let the required amount of cereal be x_1 grams, and the amount of sugar substitute be x_2 grams. Then the total calorie content of the meal is

$6x_1 + 2x_2$ calories

and the protein content is

$0.1x_1 + 0.1x_2$ grams.

So if the requirements are to be met, we must have

$6x_1 + 2x_2 = 500$

and $0.1x_1 + 0.1x_2 = 10$.

We have to find the values of x_1 and x_2 which satisfy these equations simultaneously.

A systematic method for solving equations of this type will be described in Subsection 1.2. You have already had to solve pairs of equations like this in previous units, and should have no trouble in arriving at the solution

$x_1 = 75, \quad x_2 = 25$.

So John must eat 75 grams of cereal and 25 grams of sugar substitute to satisfy his requirements.

In more realistic problems the number of unknowns and of equations can easily be larger than 2. For example, if John had also had toast for breakfast, we should have brought a third unknown x_3 into the equations. Had he taken Vitamin C into account as well we should have got a third equation.

The rest of this section is concerned with developing a systematic method for solving sets of simultaneous equations, starting with the case of two equations in two unknowns.

1.2 Solution of two simultaneous equations in two unknowns

We should like to develop a systematic method for finding the values of x_1 and x_2 which satisfy the equations

$$\begin{array}{rcl} a_1x_1 + b_1x_2 & = & c_1 \\ a_2x_1 + b_2x_2 & = & c_2 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

where the a s, b s and c s are given quantities.

Simultaneous equations have one very useful property, which we use time and time again. If we add or subtract any multiple of one equation to or from another, then the values of x_1 and x_2 which satisfy the original equations will satisfy the new one as well. (If you have studied *M203*, you may remember seeing a proof of this.) We use this property to obtain a new equation with no term in the first unknown x_1 .

M203, Unit 11

Example 1

Subtract a multiple of E_1 from E_2 to obtain an equation which has no term in x_1 from the simultaneous equations

$$\begin{array}{rcl} x_1 + 2x_2 & = & 5 \\ 2x_1 + 7x_2 & = & 16. \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

If we double the first equation, we get

$$2x_1 + 4x_2 = 10. \quad 2E_1$$

If we now subtract this from E_2 , we get the equation

$$3x_2 = 6. \quad E_2 - 2E_1$$

The advantage of this new equation is that it is simpler than the ones we started with. If we now choose to solve the simultaneous equations

$$\begin{array}{rcl} x_1 + 2x_2 & = & 5 \\ 3x_2 & = & 6 \end{array} \quad \begin{array}{l} E_1 \\ E_{2a} \end{array}$$

- (i) we have a much simpler task,
- (ii) the solution will be the same as if we had solved E_1 and E_2 .

We shall call this new equation E_{2a} , as for solution purposes it replaces E_2 .

These new equations are labelled $2E_1$ and $E_2 - 2E_1$ because that is how they are obtained from E_1 and E_2 .

Example 2

Solve the simultaneous equations of Example 1:

$$\begin{array}{rcl} x_1 + 2x_2 & = & 5 \\ 2x_1 + 7x_2 & = & 16. \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

This is a two-stage process.

Stage 1 is the process described in Example 1. The process of subtracting a multiple of E_1 from E_2 to obtain an equation E_{2a} without a term in x_1 is called the **elimination stage**.

Stage 2 consists of solving the simplified set of equations found in Stage 1:

$$\begin{array}{rcl} x_1 + 2x_2 & = & 5 \\ 3x_2 & = & 6. \end{array} \quad \begin{array}{l} E_1 \\ E_{2a} \end{array}$$

Now E_{2a} is just a simple equation in one unknown, which we solve to get

$$x_2 = 2.$$

Substituting this value into E_1 , we get

$$x_1 + 4 = 5$$

so that $x_1 = 1$. This gives us the solution

$$x_1 = 1, \quad x_2 = 2.$$

Stage 2 is called **back substitution** as, to obtain the solution, we always start with the bottom equation and work *back* through the equations, substituting the values we have found into the equation above, in order to find the next unknown.

It is always advisable to check your solution by substituting the values that you have found back into the original equations. For instance, checking the example above,

$$\text{LHS of } E_1: x_1 + 2x_2 = 1 + (2 \times 2) = 5 = \text{RHS.}$$

$$\text{LHS of } E_2: 2x_1 + 7x_2 = (2 \times 1) + (7 \times 2) = 16 = \text{RHS.}$$

LHS and RHS mean 'left hand side' and 'right hand side'.

Exercise 1

Use the method above to solve the simultaneous equations

$$x_1 - 2x_2 = 5$$

 E_1

$$2x_1 + 3x_2 = -4$$

 E_2

and check your solution.

(Solution on p. 53)

It is not difficult to see how to convert this method into a systematic rule for solving any pair of simultaneous linear equations in two unknowns. The general form of such a pair is

$$a_1x_1 + b_1x_2 = c_1$$

 E_1

$$a_2x_1 + b_2x_2 = c_2$$

 E_2

and all we need to do is to decide what multiple of E_1 to subtract from E_2 to obtain an equation with no term in x_1 .

If we multiply E_1 by the **multiplier** a_2/a_1 , we get the equation

$$a_2x_1 + \left(\frac{a_2}{a_1}\right)b_1x_2 = \left(\frac{a_2}{a_1}\right)c_1.$$

If we now subtract this from E_2 , this will provide us with an equation with no term in x_1 . Since we want to do this process automatically, there would be difficulties if a_1 were zero, as we would find ourselves (or the computer) trying to divide by zero. You can see the way that we deal with this difficulty in Section 3.

Rule for obtaining the equation E_{2a}

If

$$a_1x_1 + b_1x_2 = c_1$$

 E_1

$$a_2x_1 + b_2x_2 = c_2$$

 E_2

and a_1 is not zero,

then $E_2 - \left(\frac{a_2}{a_1}\right)E_1$ will be an equation without a term in x_1 , which we label E_{2a} .

This rule should help you with the following example and exercise.

Example 3

Solve the simultaneous equations

$$2x_1 + 3x_2 = 3$$

 E_1

$$3x_1 + 2x_2 = -\frac{1}{2}$$

 E_2

Stage 1: using the rule above, we know that $E_2 - \frac{3}{2}E_1$ will be an equation without a term in x_1 .

$E_2 - \frac{3}{2}E_1$ gives us

$$(3x_1 + 2x_2) - \frac{3}{2}(2x_1 + 3x_2) = -\frac{1}{2} - \left(\frac{3}{2} \times 3\right),$$

which simplifies to

$$-\frac{5}{2}x_2 = -5. \quad E_{2a}$$

Stage 2: solve the equations

$$2x_1 + 3x_2 = 3 \quad E_1$$

$$-\frac{5}{2}x_2 = -5. \quad E_{2a}$$

From E_{2a} , $x_2 = 2$. Substituting this value into E_1 , we get

$$2x_1 + (3 \times 2) = 3.$$

Hence $x_1 = -\frac{3}{2}$. This gives a solution

$$x_1 = -\frac{3}{2}, \quad x_2 = 2.$$

Check

LHS of E_1 : $2x_1 + 3x_2 = (2 \times (-\frac{3}{2})) + (3 \times 2) = 3 = \text{RHS}$.

LHS of E_2 : $3x_1 + 2x_2 = (3 \times (-\frac{3}{2})) + (2 \times 2) = -\frac{1}{2} = \text{RHS}$.

Exercise 2

Solve the simultaneous equations

$$5x_1 + 2x_2 = 20 \quad E_1$$

$$7x_1 - 3x_2 = -1 \quad E_2$$

(Solution on p. 53)

1.3 Solution of three simultaneous equations in three unknowns

In this subsection, we extend the technique used in Subsection 1.2 to deal with three simultaneous equations in three unknowns. As before, the process falls into two stages. In the elimination stage, multiples of E_1 are subtracted from both E_2 and E_3 . The two new equations formed (E_{2a} and E_{3a}) both have no terms in x_1 , and contain only two unknowns. To complete the elimination stage, we treat these two equations as before, finishing up with an equation (E_{3b}) in just one unknown.

Example 4

Solve the simultaneous equations

$$2x_1 + 4x_2 - 3x_3 = 16 \quad E_1$$

$$3x_1 - 2x_2 + 6x_3 = 13 \quad E_2$$

$$8x_1 - 2x_2 - 9x_3 = -2. \quad E_3$$

Stage 1. The elimination stage:

Stage 1(a): first we subtract multiples of E_1 from E_2 and E_3 to get two new equations without terms in x_1 .

$E_2 - \frac{3}{2}E_1$ gives

$$-8x_2 + \frac{21}{2}x_3 = -11 \quad E_{2a}$$

and $E_3 - \frac{8}{2}E_1$ gives

$$-18x_2 + 3x_3 = -66. \quad E_{3a}$$

Stage 1(b): we forget about the original equations for the time being. E_{2a} and E_{3a} are simply two equations in two unknowns. So, using these two, we subtract a multiple of E_{2a} from E_{3a} to obtain an equation E_{3b} without a term in x_2 .

$E_{3a} - (-\frac{18}{8})E_{2a}$ gives

$$-\frac{165}{8}x_3 = -\frac{165}{4} \quad E_{3b}$$

At this stage, the elimination process is finished; we have brought our equations to what we call the 'upper triangular' form:

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 16 \\ -8x_2 + \frac{21}{2}x_3 &= -11 \\ -\frac{165}{8}x_3 &= -\frac{165}{4} \end{aligned}$$

$$\begin{aligned} E_1 \\ E_{2a} \\ E_{3b} \end{aligned}$$

Notice that the aim is to get successive equations each with one less unknown.

Stage 2. The back substitution stage:

We can now find the values of x_1 , x_2 and x_3 from E_1 , E_{2a} and E_{3b} . Starting with E_{3b} , we get

$$x_3 = 2.$$

Substituting this value into E_{2a} , we get

$$\begin{aligned} -8x_2 + \left(\frac{21}{2} \times 2\right) &= -11, \\ x_2 &= 4. \end{aligned}$$

Finally, substituting both the values for x_2 and x_3 into E_1 , we get

$$2x_1 + (4 \times 4) - (2 \times 2) = 16.$$

Hence $x_1 = 3$, giving the solution

$$x_1 = 3, \quad x_2 = 4, \quad x_3 = 2.$$

Checking the solution, we get

$$\text{LHS of } E_1: 2x_1 + 4x_2 - 3x_3 = (2 \times 3) + (4 \times 4) - (3 \times 2) = 16 = \text{RHS.}$$

$$\text{LHS of } E_2: 3x_1 - 2x_2 + 6x_3 = (3 \times 3) - (2 \times 4) + (6 \times 2) = 13 = \text{RHS.}$$

$$\text{LHS of } E_3: 8x_1 - 2x_2 - 9x_3 = (8 \times 3) - (2 \times 4) - (9 \times 2) = -2 = \text{RHS.}$$

The process of solving simultaneous equations described in Example 4 is often referred to as the **Gaussian elimination method**, although strictly speaking the elimination is finished at Stage 1(b). The method, albeit tedious, provides a systematic approach to the problem of solving simultaneous equations, which should cope with the rather larger sets of equations which can occur in real-life applications.

Exercise 3

Solve the following simultaneous equations by Gaussian elimination:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 1 & E_1 \\ 5x_1 + x_2 + 2x_3 &= 6 & E_2 \\ 4x_1 - 2x_2 - 3x_3 &= 3. & E_3 \end{aligned}$$

(Solution on p. 53)

1.4 Organization of the work

When using Gaussian elimination to solve the equations

$$\begin{aligned} 2x_1 + x_2 &= 3 & E_1 \\ 3x_1 + 2x_2 &= 5 & E_2 \end{aligned}$$

the new equation E_{2a} is obtained by subtracting $\frac{3}{2}E_1$ from E_2 . It is worth noting that the work is done on the numbers multiplying x_1 and x_2 (called the **coefficients** of x_1 and x_2). For instance, the operation

$$2x_2 - \frac{3}{2}x_2 = \frac{1}{2}x_2$$

could be recorded as

$$2 - \frac{3}{2} = \frac{1}{2}$$

(so long as we remember that the operation was attached to x_2 !). While the elimination is being done, it is sufficient just to record the coefficients of x_1 and x_2 and the right-hand side of the equations rather than the whole equation each time. For instance, we can record the important features of E_1 and E_2 above in the number block

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & 2 & 5 \end{array} \right].$$

The first row contains the coefficients of x_1 and x_2 and the right-hand side of E_1 , and the second row contains similar information about E_2 . Alternatively, the first and second columns represent the coefficients of x_1 and x_2 , and the column after the bar represents the right-hand sides of the equations.

Often, in mathematics, information is stored in rectangular number blocks in this way. A block like this is called a **matrix**, and the numbers in it, its **elements**.

Exercise 4

Record the essential information about the following equations in matrix form.

$$3x_1 - 5x_2 = 8$$

 E_1

$$4x_1 + 7x_2 = 11$$

 E_2

(Solution on p. 53)

Example 5

Solve the simultaneous equations

$$2x_1 + x_2 = 3$$

 E_1

$$3x_1 + 2x_2 = 5$$

 E_2

This time, we shall use the matrix representing these equations throughout the elimination procedure. For brevity, we shall use $\mathbf{R}_1$ for the first row of the matrix, and so on.

The original equations can be represented by the matrix

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & 2 & 5 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \end{array}$$

Now E_{2a} is found by subtracting $\frac{3}{2}E_1$ from E_2 . Arithmetically, this is the same operation as subtracting $\frac{3}{2}\mathbf{R}_1$ from $\mathbf{R}_2$. It is useful to record this operation in another matrix. The way in which the new rows are formed can be recorded on the left of the matrix:

$$\mathbf{R}_2 - \frac{3}{2}\mathbf{R}_1 \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \end{array}$$

The important thing to note is that subtracting $\frac{3}{2}E_1$ from E_2 produces the same coefficients as the numbers we store when subtracting $\frac{3}{2}\mathbf{R}_1$ from $\mathbf{R}_2$. To operate on the rows of the matrix in this way is less tedious than writing out the equations each time.

Once the elimination has been done, the back substitution can be performed in the same way as before. Writing out the equations represented by the last matrix, we get

$$2x_1 + x_2 = 3$$

 $E_1 \text{ (from } \mathbf{R}_1)$

$$\frac{1}{2}x_2 = \frac{1}{2}$$

 $E_{2a} \text{ (from } \mathbf{R}_{2a})$

so that, from E_{2a} ,

$$x_2 = 1.$$

Substituting this into E_1 , we get

$$2x_1 + 1 = 3.$$

Hence $x_1 = 1$, giving the solution

$$x_1 = 1, \quad x_2 = 1.$$

Checking this solution,

$$\text{LHS of } E_1: \quad 2x_1 + x_2 = (2 \times 1) + 1 = 3 = \text{RHS.}$$

$$\text{LHS of } E_2: \quad 3x_1 + 2x_2 = (3 \times 1) + (2 \times 1) = 5 = \text{RHS.}$$

Exercise 5

Use the matrix method to solve the simultaneous equations

$$\begin{array}{rcl} x_1 + 2x_2 & = & 4 \\ 3x_1 - x_2 & = & 5 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

(Solution on p. 53)

Example 6

Solve the simultaneous equations of Example 4 using the matrix method. In matrix form, these give

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & 16 \\ 3 & -2 & 6 & 13 \\ 8 & -2 & -9 & -2 \end{array} \right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Stage 1. The elimination stage:

Each part of the elimination stage in Example 4 has an equivalent part in matrix form.

Stage 1(a)

$$\begin{array}{l} R_2 - \frac{3}{2}R_1 \\ R_3 - \frac{8}{2}R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & -3 & 16 \\ 0 & -8 & \frac{21}{2} & -11 \\ 0 & -18 & 3 & -66 \end{array} \right] \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

Stage 1(b)

$$R_{3a} - \left(-\frac{18}{8}\right)R_{2a} \left[\begin{array}{ccc|c} 2 & 4 & -3 & 16 \\ 0 & -8 & \frac{21}{2} & -11 \\ 0 & 0 & -\frac{165}{8} & -\frac{165}{4} \end{array} \right] \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

Note the triangle of zeros in the matrix which appears at the end of the elimination. This shows us that each equation has one less unknown than the previous one, which is what we need to be able to do the back substitution.

Stage 2. The back substitution stage:

This is exactly the same as Stage 2 of Example 4.

Exercise 6

Rewrite the solution of Exercise 3 in matrix form.

(Solution on p. 53)

Summary of Section 1

In this section, we have seen a method for solving a set of **linear simultaneous equations**, called **Gaussian elimination**.

The method has two stages:

Stage 1. The elimination stage: this consists of systematically reducing the number of unknowns in each successive equation until we are left with a set of equations whose matrix representation has a triangle of zeros in the bottom left-hand corner. The method used is shown below.

To obtain an equation E_{2a} without a term in x_1 from the simultaneous equations

$$\begin{array}{rcl} a_1x_1 + b_1x_2 + c_1x_3 & = & d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 & = & d_2 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

(where the a s, b s, c s and d s are given, and $a_1 \neq 0$), subtract $\left(\frac{a_2}{a_1}\right)E_1$ from E_2 . The number $\frac{a_2}{a_1}$ is called a **multiplier**.

Stage 2. The back substitution stage: this consists of finding the unknowns x_1 , x_2 and x_3 by solving the reduced set of equations which we obtained at the end of the elimination. These will have the form

$$\begin{array}{rcl} a_1x_1 + b_1x_2 + c_1x_3 & = & d_1 \\ b_2^*x_2 + c_2^*x_3 & = & d_2^* \\ c_3^*x_3 & = & d_3^* \end{array} \quad \begin{array}{l} E_1 \\ E_{2a} \\ E_{3b} \end{array}$$

- (i) Find x_3 from E_{3b} .
- (ii) Substitute this value of x_3 into E_{2a} , and hence find x_2 .
- (iii) Substitute the values of x_3 and x_2 into E_1 , and hence find x_1 .

The advantage of this method is that it is systematic, and can easily be extended to larger sets of simultaneous equations. The calculations can be simplified by using matrix notation.

End of section exercises

Solve the following sets of simultaneous equations using Gaussian elimination. Use a matrix to represent the equations for the elimination stage, and check your solution in each case.

Exercise 7

- (i) $\begin{array}{r} x_1 - 2x_2 = 5 \\ 4x_1 + x_2 = 2 \end{array}$
- (ii) $\begin{array}{r} 2x_1 - 3x_2 = 16 \\ 3x_1 - 4x_2 = 22 \end{array}$

Exercise 8

- (i) $\begin{array}{r} x_1 - 3x_2 + x_3 = 11 \\ 2x_1 + x_2 + 4x_3 = 3 \\ 3x_1 - 5x_2 - 7x_3 = 11 \end{array}$
- (ii) $\begin{array}{r} 3x_1 - 2x_2 + 5x_3 = 10 \\ x_1 + 2x_2 + 3x_3 = 10 \\ x_1 + 5x_2 - 3x_3 = 10 \end{array}$

(Solutions on p. 53)

2 Types of solution

2.1 A geometric look at different types of solution

(a) Linear equations in two unknowns

The term 'linear' comes from the fact that the graphical representation of an equation of the form

$$ax + by = c$$

is a straight line.

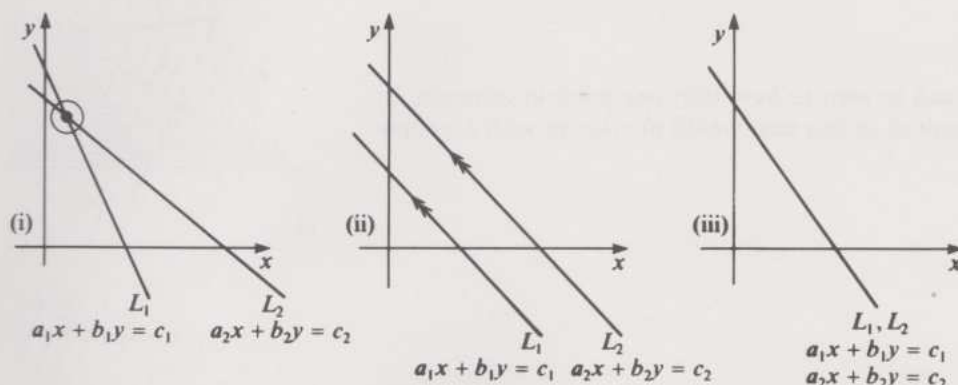
For the purposes of this subsection, we shall refer to the unknown quantities as x , y and z (rather than x_1 , x_2 and x_3), as these letters are traditionally used for three-dimensional Cartesian coordinates.

When we write down two linear simultaneous equations in two unknowns

$$\begin{array}{rcl} a_1x + b_1y & = & c_1 \\ a_2x + b_2y & = & c_2 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

these can be represented graphically in two dimensions by two straight lines L_1 and L_2 . All the points on L_1 satisfy E_1 , and all the points on L_2 satisfy E_2 . Hence the intersection of the lines is a point satisfying both E_1 and E_2 .

The solution of the simultaneous equations can be thought of graphically as the coordinates of the point of intersection of the two lines L_1 and L_2 . If we draw two arbitrary lines, there are three situations which can arise, which are illustrated below.



(i) The most likely situation to occur is that the lines will not be parallel, and will have one point in common. The coordinates of this point will also be the *unique solution* to the simultaneous equations.

(ii) The situation when the lines are parallel occurs much less often. In this case, the lines don't intersect at all, and the simultaneous equations will have *no solution*.

(iii) The least likely situation to occur is that the two lines coincide. In this case, all the coordinates of the points lying on the line will satisfy the equation of the line, and the simultaneous equations will have an *infinite number of solutions*.

So far, we have only met simultaneous equations which have a unique solution. We shall be looking at the kind of equations which have no solution, or which have an infinite number of solutions, in the tape subsection, Subsection 2.2.

Exercise 1

Sketch the graphs of the following pairs of simultaneous equations.

- | | | |
|-----------------|------------------|-------------------|
| (i) $x + y = 4$ | (ii) $x + y = 4$ | (iii) $x + y = 4$ |
| $2x - 3y = 3$ | $2x + 2y = 5$ | $2x + 2y = 8$ |

(The 'solutions' to these pairs of equations will be discussed in the tape subsection, Subsection 2.2.)

(Solutions on p. 54)

(b) Linear equations in three unknowns

Again a linear equation in three unknowns

$$ax + by + cz = d$$

can be represented geometrically in three dimensions by a plane. If we consider three equations

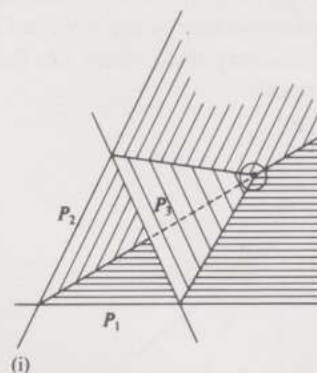
$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 & E_1 \\ a_2x + b_2y + c_2z &= d_2 & E_2 \\ a_3x + b_3y + c_3z &= d_3 & E_3 \end{aligned}$$

these could also be thought of as the equations of three planes P_1 , P_2 and P_3 .

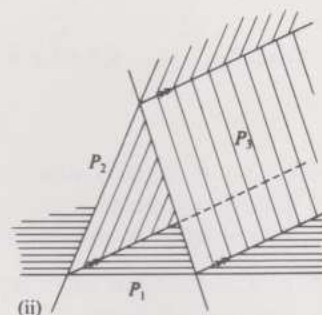
By an argument similar to that of the two-dimensional case, the solution of three such simultaneous equations is the coordinates of the point of intersection of the three planes P_1 , P_2 and P_3 . Now, there are more ways of arranging three planes than there were of arranging two lines in two dimensions. Three of the possibilities are given over the page.

(i) *A unique solution*

The planes P_1 , P_2 and P_3 can be seen to have only *one* point in common. In other words, the three equations in this case would provide us with a unique solution.

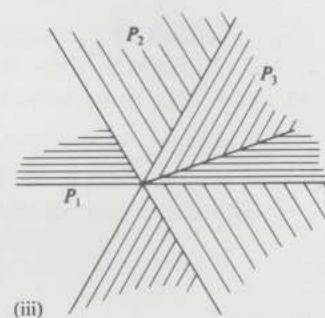
(ii) *No solution*

The planes P_1 , P_2 and P_3 clearly have no points in common. In this case the three equations would have no solution.

(iii) *An infinite number of solutions*

The figure opposite illustrates that the three planes could intersect in a common line.

In this case the equations represented by the planes would have an infinite number of solutions, namely the points on the common line of intersection of the planes.



Even if all the *geometric* possibilities are not exhausted here, they all correspond to one of the *algebraic* possibilities listed below.

If we have n equations in n unknowns, then *either*

(i) the equations will have a unique solution (which is the most frequent occurrence) *or*

(ii) the equations will have no solution (which is rare) *or*

(iii) the equations will have an infinite number of solutions (very rare).

In the following Tape Subsection, we shall see how to determine the type of solution using the elimination procedure on any given set of n simultaneous equations in n unknowns.

2.2 Finding the type of solution algebraically (Tape)

Start the tape now.



Another geometric possibility is for all three planes to coincide. Can you think of any more?

SET 1

Solve the simultaneous equations

$$x_1 + x_2 = 4 \quad E_1$$

$$2x_1 - 3x_2 = 3 \quad E_2$$

The matrix representing these equations is

$$\left[\begin{array}{cc|c} & & \\ & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_2 \end{array}$$

Stage 1: the matrix after the elimination is

$$\left[\begin{array}{cc|c} & & \\ 0 & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \end{array}$$

Stage 2: the equations represented by this matrix are

$$\begin{array}{l} E_1 \\ E_{2a} \end{array}$$

and back substitution gives

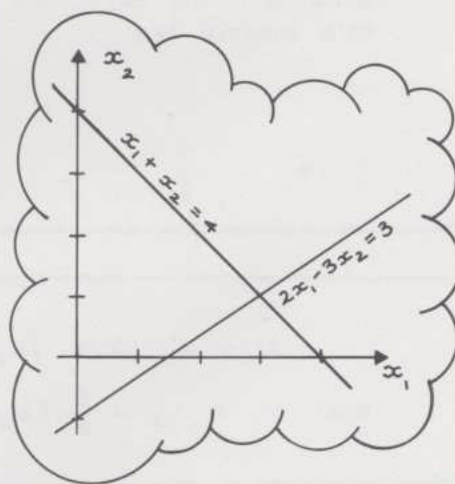
$$\text{from } E_{2a}, \quad x_2 =$$

$$\text{from } E_1, \quad x_1 =$$

Checking solution in the original equations

$$x_1 + x_2 =$$

$$2x_1 - 3x_2 =$$



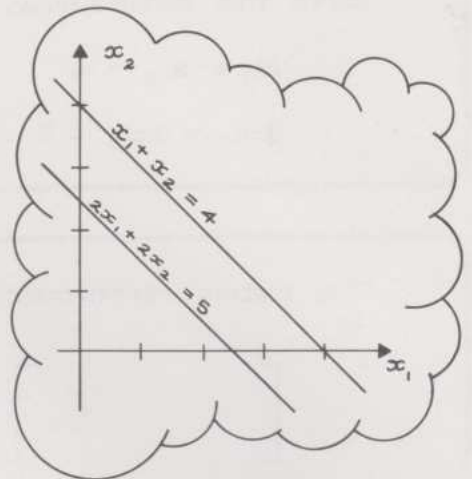
A UNIQUE SOLUTION

SET 2

Solve the simultaneous equations

$$x_1 + x_2 = 4 \quad E_1$$

$$2x_1 + 2x_2 = 5 \quad E_2$$



The matrix representing these equations is

$$\left[\begin{array}{cc|c} & & \\ & & \end{array} \right] \begin{array}{l} R_1 \\ R_2 \end{array}$$

Stage 1: the matrix after the elimination is

$$\left[\begin{array}{cc|c} & & \\ & & \end{array} \right] \begin{array}{l} R_1 \\ R_{2a} \end{array}$$

Stage 2: the equations represented by this matrix are

$$E_1$$

$$E_{2a}$$

$$\left. \begin{array}{l} x_1 + x_2 = 4 \quad (E_1) \\ \text{and } x_1 + x_2 = \frac{5}{2} \quad (E_2 \div 2) \end{array} \right\}$$

Equations inconsistent

NO SOLUTION

SET 3

Solve the simultaneous equations

$$x_1 + x_2 = 4 \quad E_1$$

$$2x_1 + 2x_2 = 8 \quad E_2$$

The matrix representing these equations is

$$\left[\begin{array}{cc|c} & & R_1 \\ & & R_2 \end{array} \right]$$

Stage 1: the matrix after the elimination is

$$\left[\begin{array}{cc|c} & & R_1 \\ 0 & & R_2 \end{array} \right]$$

Stage 2: the equations represented by this matrix are

$$E_1$$

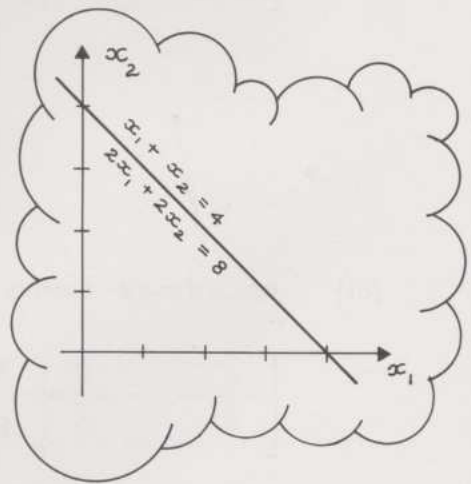
$$E_2 \alpha$$

With only one equation

let $x_2 = k$, then

from E_1 , $x_1 =$

where k is an arbitrary number.



Checking solution in original equations,

$$x_1 + x_2 =$$

$$2x_1 + 2x_2 =$$

AN INFINITE NUMBER OF SOLUTIONS

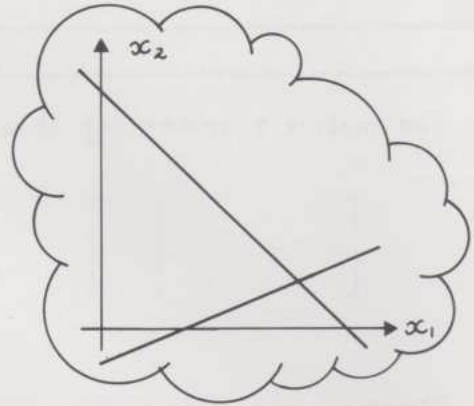
SET 4

Summary

A set of simultaneous equations can have

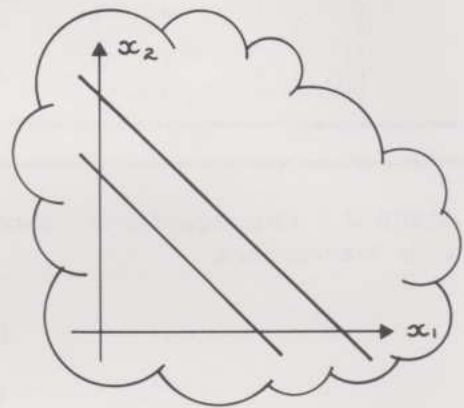
- (i) a unique solution

$$\left[\begin{array}{cc|c} \checkmark & * & * \\ 0 & \checkmark & * \end{array} \right]$$



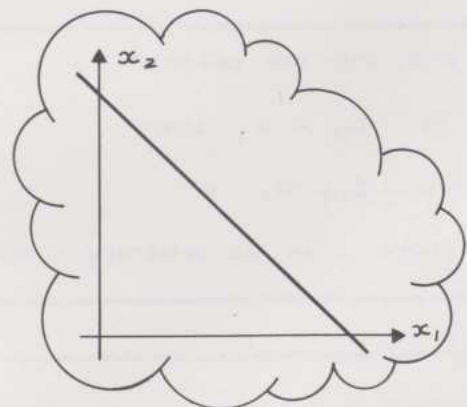
- (ii) no solution (when the equations are inconsistent)

$$\left[\begin{array}{cc|c} \checkmark & * & * \\ 0 & 0 & \checkmark \end{array} \right]$$



- (iii) an infinite number of solutions

$$\left[\begin{array}{cc|c} \checkmark & * & * \\ 0 & 0 & 0 \end{array} \right]$$

Key

- $\checkmark$: must be non-zero
 $*$: can be anything

SET 5

Solve the simultaneous equations

$$x_1 + x_2 - x_3 = 3 \quad E_1$$

$$2x_1 - 3x_2 - x_3 = 0 \quad E_2$$

$$3x_1 + 7x_2 + 2x_3 = 8 \quad E_3$$

The matrix representing these equations is

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

stage 1: the elimination

$$(a) \quad \left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

$$(b) \quad \left[\begin{array}{ccc|c} & & & \\ \circ & & & \\ \circ & \circ & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

stage 2: the equations represented by this matrix are

$$\begin{array}{l} E_1 \\ E_{2a} \\ E_{3b} \end{array}$$

The back substitution:

$$\text{from } E_{3b} \quad x_3 =$$

$$\text{from } E_{2a} \quad x_2 =$$

$$\text{from } E_1 \quad x_1 =$$

Checking the solution in E_3

$$3x_1 + 7x_2 + 2x_3 =$$

SET 6

Solve the simultaneous equations

$$\begin{array}{rcl} x_1 - 2x_2 + 3x_3 & = & 0 \quad E_1 \\ 2x_1 + 3x_2 - x_3 & = & 4 \quad E_2 \\ x_1 - 23x_2 + 24x_3 & = & 18 \quad E_3 \end{array}$$

The matrix representing these equations is

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Stage 1: the elimination

$$(a) \left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

$$(b) \left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

Stage 2: the equation E_{3b} represented by this matrix is

$$E_{3b}$$

These equations are inconsistent and have no solution.

SET 7

Solve the simultaneous equations

$$x_1 + x_2 + x_3 = 3 \quad E_1$$

$$x_1 + 2x_2 + 3x_3 = 6 \quad E_2$$

$$2x_1 - x_2 - 4x_3 = -3 \quad E_3$$

The matrix representing these equations is

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

stage 1: the elimination

$$(a) \quad \left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

$$(b) \quad \left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array} \right] \quad \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

stage 2: the equations represented by this matrix are

$$E_1$$

$$E_{2a}$$

The back substitution:

let $x_3 = k$ so

from E_{2a} $x_2 =$

from E_1 $x_1 =$

where k is arbitrary.

Checking this solution in E_3

$$2x_1 - x_2 - 4x_3 =$$

SET 8

Summary

The matrix formed at the end of the elimination will usually have one of the following characteristics :

$$(i) \left[\begin{array}{ccc|c} \checkmark & * & * & * \\ 0 & \checkmark & * & * \\ 0 & 0 & \checkmark & * \end{array} \right]$$

UNIQUE SOLUTION

$$(ii) \left[\begin{array}{ccc|c} \checkmark & * & * & * \\ 0 & \checkmark & * & * \\ 0 & 0 & 0 & \checkmark \end{array} \right]$$

NO SOLUTION (Equations inconsistent)

$$(iii) \left[\begin{array}{ccc|c} \checkmark & * & * & * \\ 0 & \checkmark & * & * \\ 0 & 0 & 0 & 0 \end{array} \right]$$

AN INFINITE NUMBER OF SOLUTIONS

Key

✓ : must be non-zero

* : can be anything

Should you need any more practice at determining types of solutions, try the following exercise.

Exercise 2

Determine the type of solution of each of the following sets of equations. Find the solution where possible. In the case of the set of equations with an infinite number of solutions, find the general solution.

(i)	$2x_1 - x_2 - x_3 = 8$	E_1
	$3x_1 - x_2 - 5x_3 = 6$	E_2
	$x_1 + x_2 - 11x_3 = 2$	E_3
(ii)	$x_1 - 2x_2 + 4x_3 = 7$	E_1
	$2x_1 + 4x_2 - x_3 = -3$	E_2
	$x_1 + 14x_2 - 14x_3 = -27$	E_3
(iii)	$3x_1 - 2x_2 + 3x_3 = 2$	E_1
	$2x_1 + 3x_2 - 5x_3 = 4$	E_2
	$4x_1 + 2x_2 + x_3 = 1$	E_3

(Solutions on p. 54)

2.3 Linear dependence

Before discussing this topic, it is convenient to introduce some new notation. Suppose we have a set of equations

$$\begin{array}{rcl} a_1x_1 + b_1x_2 + c_1x_3 & = & d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 & = & d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 & = & d_3 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

Sometimes we want to refer to the matrix of the left-hand side coefficients, sometimes to the matrix of the numbers on the right-hand side of the equations, and sometimes to the whole matrix. In future, we will refer to the matrix of the left-hand side coefficients as $\mathbf{A}$; that is,

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Similarly, we will refer to the matrix of the right-hand sides of the equations as $\mathbf{b}$; that is

$$\mathbf{b} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

This is a matrix with one column.

Finally, we shall refer to the matrix which contains both sets of information as $\mathbf{A}|\mathbf{b}$; that is,

$$\mathbf{A}|\mathbf{b} = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right].$$

This of course is the matrix which we use to do the elimination.

We saw in Subsection 2.2 that it is possible to tell what type of solution a set of simultaneous equations will have, by inspection of the matrix obtained after the elimination. The following examples examine the elimination procedure in more detail to see how a row of zeros might occur.

Example 1

If you have to solve any set of simultaneous equations—for example

$$\begin{array}{rcl} x_1 + x_2 - x_3 & = & 3 \\ 2x_1 - 3x_2 - x_3 & = & 0 \\ 3x_1 + 7x_2 + 2x_3 & = & 8 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

the first thing you do is to write down the matrix $\mathbf{A}|\mathbf{b}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & -3 & -1 & 0 \\ 3 & 7 & 2 & 8 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Then you go through (in the case of three equations in three unknowns) the two-stage elimination procedure:

Stage 1(a)

$$\begin{array}{l} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - 3\mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -5 & 1 & -6 \\ 0 & 4 & 5 & -1 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b)

$$\mathbf{R}_{3a} - (-\frac{4}{5})\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & \frac{29}{5} & -\frac{29}{5} \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

At Stage 1(b), inspection of the matrix tells you that you are going to have a unique solution, as there is no problem with the back substitution.

Let us look more carefully at how we obtained $\mathbf{R}_{3b}$. We know from Stage 1(b) that

$$\mathbf{R}_{3b} = \mathbf{R}_{3a} + \frac{4}{5}\mathbf{R}_{2a}. \quad (1)$$

The equals sign in this context means 'is made up of'.

But we also know from Stage 1(a) that

$$\mathbf{R}_{2a} = \mathbf{R}_2 - 2\mathbf{R}_1,$$

$$\mathbf{R}_{3a} = \mathbf{R}_3 - 3\mathbf{R}_1.$$

If we substitute these expressions for $\mathbf{R}_{2a}$ and $\mathbf{R}_{3a}$ into (1), we get

$$\begin{aligned} \mathbf{R}_{3b} &= \mathbf{R}_3 - 3\mathbf{R}_1 + \frac{4}{5}(\mathbf{R}_2 - 2\mathbf{R}_1) \\ &= -\frac{23}{5}\mathbf{R}_1 + \frac{4}{5}\mathbf{R}_2 + \mathbf{R}_3. \end{aligned}$$

You can see that, independent of the particular elimination we have done, $\mathbf{R}_{3b}$ is always given by an expression of the form

$$\mathbf{R}_{3b} = k_1\mathbf{R}_1 + k_2\mathbf{R}_2 + k_3\mathbf{R}_3$$

(where the k s are numbers—not all zero).

So $\mathbf{R}_{3b}$ is a *linear combination* of the rows $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ of the original matrix $\mathbf{A}|\mathbf{b}$. You have met linear combinations before, in Unit 6. In fact, each new row we form during the elimination process is made up of a linear combination of the rows of $\mathbf{A}|\mathbf{b}$. For instance, in the last example,

$$\mathbf{R}_{2a} = -2\mathbf{R}_1 + \mathbf{R}_2 + 0\mathbf{R}_3.$$

Sometimes, one of these linear combinations may result in a row of zeros, as the following example will demonstrate.

Example 2

If we go through the same procedure as in the last example with the equations

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 3 \\ x_1 + 2x_2 + 3x_3 & = & 6 \\ 2x_1 - x_2 - 4x_3 & = & -3 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

the elimination goes as follows:

Stage 1(a)

$$\begin{array}{l} \mathbf{R}_2 - \mathbf{R}_1 \\ \mathbf{R}_3 - 2\mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -6 & -9 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b)

$$\mathbf{R}_{3a} - (-3)\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

The row of zeros tells us that this set of equations will give us an infinite number of solutions.

This time, we know that

$$\mathbf{R}_{3b} = \mathbf{R}_{3a} + 3\mathbf{R}_{2a}$$

where

$$\mathbf{R}_{2a} = \mathbf{R}_2 - \mathbf{R}_1$$

and

$$\mathbf{R}_{3a} = \mathbf{R}_3 - 2\mathbf{R}_1,$$

giving

$$\mathbf{R}_{3b} = -5\mathbf{R}_1 + 3\mathbf{R}_2 + \mathbf{R}_3.$$

So we know that for this set of equations,

$$-5\mathbf{R}_1 + 3\mathbf{R}_2 + \mathbf{R}_3 = \mathbf{0} \quad (\text{a row of zeros}).$$

It can be shown that for any equations which have an infinite number of solutions, there will be some linear combination of the rows of $\mathbf{A}|\mathbf{b}$ which is equal to a row of zeros:

$$k_1\mathbf{R}_1 + k_2\mathbf{R}_2 + k_3\mathbf{R}_3 = \mathbf{0}$$

(where the k s are numbers, not all zero).

When such a relationship exists between the rows of a matrix, we say that the rows are **linearly dependent**.

Rows $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_n$ of a matrix are said to be **linearly dependent** if some linear combination is equal to a zero row:

$$k_1\mathbf{R}_1 + k_2\mathbf{R}_2 + \dots + k_n\mathbf{R}_n = \mathbf{0}$$

where the k s are numbers, not all zero.

If the rows are not linearly dependent, we say that they are **linearly independent**. Again, you have already met the concept of linear independence in Unit 6.

To complete the picture, let us have a look at the case of a set of simultaneous equations which have no solution.

Example 3

If we do the elimination procedure for the equations

$$\begin{array}{rcl} x_1 - 2x_2 + 3x_3 & = & 0 \\ 2x_1 + 3x_2 - x_3 & = & 4 \\ x_1 - 23x_2 + 24x_3 & = & 18 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

we get

Stage 1(a)

$$\begin{array}{l} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - \mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -7 & 4 \\ 0 & -21 & 21 & 18 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b)

$$\mathbf{R}_{3a} - \left(\frac{-21}{7}\right)\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -7 & 4 \\ 0 & 0 & 0 & 30 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

It is the row (0, 0, 0, 30) which tells us that these equations are inconsistent and have no solution. Now, this time (as for most other sets of inconsistent equations), we can see that

- (i) the rows of $\mathbf{A}$ are linearly dependent but
- (ii) the rows of $\mathbf{A}|\mathbf{b}$ are linearly independent.

A certain amount of common sense has to be exercised when interpreting this type of solution. For instance, the equations

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 1 \\ 2x_1 + 2x_2 + 2x_3 & = & 2 \\ 2x_1 + 2x_2 + 2x_3 & = & 4 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

give us the following matrix after elimination:

$$\begin{array}{l} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - 2\mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

This is a rare case where you can see both the row of zeros that you would expect to see for an infinite number of solutions ($\mathbf{R}_{2a}$), and also the situation that you expect to find for inconsistency ($\mathbf{R}_{3a}$). Faced with both these possibilities, the inconsistency must take priority, as at no time in the process of solving these equations is it possible to obtain an equation

$$0x_1 + 0x_2 + 0x_3 = 2$$

without the equations being inconsistent. We can express this in terms of the ideas of this subsection as follows: if a certain linear combination of the rows of $\mathbf{A}$ is equal to zero, and *the same* linear combination of the rows of $\mathbf{A}|\mathbf{b}$ is non-zero, then the equations are inconsistent. On the other hand, if this situation is not present, then any linear dependency guarantees an infinite number of solutions.

We shall use these ideas of linear dependence in the next section, and again in the units on matrices and eigenvalues which you will meet in the course.

Exercise 3

The equations which you solved in Exercise 2(ii),

$$\begin{array}{rcl} x_1 - 2x_2 + 4x_3 & = & 7 \\ 2x_1 + 4x_2 - x_3 & = & -3 \\ x_1 + 14x_2 - 14x_3 & = & -27 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

have an infinite number of solutions. Use the elimination procedure to determine what linear combination of the rows of $\mathbf{A}|\mathbf{b}$ gives a row of zeros.

(Solution on p. 55)

The ideas in the last three examples can be summarized as follows.

Given a set of equations represented by the matrix $\mathbf{A}|\mathbf{b}$, as explained on p. 23, then:

- (i) if the rows of $\mathbf{A}$ are linearly independent, then the equations have a unique solution;
- (ii) if the rows of $\mathbf{A}$ are linearly dependent, and the rows of $\mathbf{A}|\mathbf{b}$ are linearly independent, then the equations are inconsistent, and have no solution;
- (iii) if the rows of $\mathbf{A}|\mathbf{b}$ are linearly dependent, then the equations usually have an infinite number of solutions.

Summary of Section 2

1. Given a set of linear simultaneous equations, they will have *either*:

- (i) a unique solution (the most common situation), *or*
- (ii) no solution (rare), *or*
- (iii) an infinite number of solutions (very rare).

2. The rows $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_n$ of a matrix are said to be **linearly dependent**, if some linear combination of these rows is zero,

i.e. constants $k_1, k_2, \dots, k_n$ (not all zero) exist such that

$$k_1\mathbf{R}_1 + k_2\mathbf{R}_2 + \dots + k_n\mathbf{R}_n = \mathbf{0} \text{ (row of zeros).}$$

3. To find out which type of solution a particular set of simultaneous equations has, do the elimination automatically on the matrix $\mathbf{A}|\mathbf{b}$. The matrix formed at the end of the elimination will usually have one of the following characteristics.

[In the following table, a matrix entry which must be non-zero is denoted by a tick ($\checkmark$), while a matrix entry which could be zero or non-zero is denoted by a star (*).]

Characteristics of matrix obtained after elimination	Type of solution	Comments
$\left[\begin{array}{ccc c} \checkmark & * & * & * \\ 0 & \checkmark & * & * \\ 0 & 0 & \checkmark & * \end{array} \right]$ no zero row of $\mathbf{A}$	unique solution	The rows of $\mathbf{A}$ are linearly independent.
$\left[\begin{array}{ccc c} \checkmark & * & * & * \\ 0 & \checkmark & * & * \\ 0 & 0 & 0 & \checkmark \end{array} \right]$ a zero row of $\mathbf{A}$ with the RHS entry non-zero	no solution (equations inconsistent)	The rows of $\mathbf{A}$ are linearly dependent, whereas the rows of $\mathbf{A} \mathbf{b}$ are not.
$\left[\begin{array}{ccc c} \checkmark & * & * & * \\ 0 & \checkmark & * & * \\ 0 & 0 & 0 & 0 \end{array} \right]$ a zero row of $\mathbf{A} \mathbf{b}$	an infinite number of solutions	The rows of $\mathbf{A} \mathbf{b}$ (and hence also of $\mathbf{A}$) are linearly dependent.

End of section exercises

In the following exercises, determine the nature of the solutions of the following sets of equations. In the case where the equations have a unique solution, find it. In the case where the equations have an infinite number of solutions:

- (i) find the general solution;
- (ii) find which linear combination of the rows of $\mathbf{A}|\mathbf{b}$ gives a row of zeros.

Exercise 4

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 7 \\ x_1 + 3x_2 - 4x_3 &= 20 \\ x_1 + 18x_2 - 31x_3 &= 40 \end{aligned}$$

$$\begin{aligned} E_1 \\ E_2 \\ E_3 \end{aligned}$$

Exercise 5

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 6 \\ x_1 + 3x_2 - 4x_3 &= 7 \\ 2x_1 + 7x_2 - 12x_3 &= 12 \end{aligned}$$

$$\begin{aligned} E_1 \\ E_2 \\ E_3 \end{aligned}$$

Exercise 6

$$\begin{aligned} 5x_1 - x_2 - x_3 &= 2 \\ x_1 - 4x_2 + x_3 &= 14 \\ 6x_1 + 14x_2 - 6x_3 &= -52 \end{aligned}$$

$$\begin{aligned} E_1 \\ E_2 \\ E_3 \end{aligned}$$

(Solutions on p. 55)

3 Some possible difficulties (Television Section)

3.1 Preparation for the television programme

In this unit we have used the systematic method of Gaussian elimination to solve sets of simultaneous equations and, to date, nothing unpleasant has happened. However, this is not always the case, as you will see in the television programme and later in this section.

The first part of the television programme (and Subsection 3.2) discusses sets of equations which are *ill-conditioned*. The equations in the first part of the following exercise relate to the 'practical' problem discussed at the beginning of the programme. The second part of the exercise is strictly speaking post-television work, but we thought you would be interested to see these two problems together.

Exercise 1

- (i) Verify that the solution of the set of equations

$$\begin{aligned} 6b + 4g + 3l &= 735 & E_1 \\ 7\frac{1}{2}b + 6g + 5l &= 1052 & E_2 \\ 10b + 7\frac{1}{2}g + 6l &= 1331 & E_3 \end{aligned}$$

is

$$b = 74, g = -18, l = 121.$$

- (ii) Verify that the solution of the set of equations

$$\begin{aligned} 6b + 4g + 3l &= 734 & E_1 \\ 7\frac{1}{2}b + 6g + 5l &= 1049\frac{1}{2} & E_2 \\ 10b + 7\frac{1}{2}g + 6l &= 1331 & E_3 \end{aligned}$$

is

$$b = 53, g = 62, l = 56.$$

(Note that the equations in (ii) are similar to those in (i), except for small changes to the right-hand sides of E_1 and E_2 .)

(No solution given.)

The second part of the television programme discusses the effect of rounding error in sets of equations which are not ill-conditioned. You may have noticed that so far in this unit, fractions rather than decimals have been used to solve the problems. This has been done to avoid introducing any error into our calculations, and each time we have obtained the *exact* solutions of the sets of equations. This is not the usual state of affairs. Normally we use a calculator or computer to help us do these problems. These machines hold approximations of numbers—albeit that they are very accurate approximations. The result is that the solution that we obtain using a computer is not exact.

The second problem in the programme is done in two ways. The first time it is done using exact arithmetic (i.e. fractions)—and the second time 'three figure arithmetic' is used. This means that *each* number at *each* stage of a calculation is rounded to three significant figures. The following example demonstrates how this arithmetic works.

Example 1

Suppose that we wish to evaluate

$$(34.87 \times 0.01854) - 8.575$$

using three figure arithmetic. First, each number is rounded to three significant figures. 34.87 is rounded to 34.9, 0.01854 is rounded to 0.0185, and 8.575 is rounded up to 8.58.



TV 9

You met ill-conditioning in a different context in *Unit 1*. The exact definition for the present context is given on page 29.

In the television programme round rather than square brackets are used for matrices. In fact both types of notation are in common use.

So, $34.87 \times 0.018\,54$ is calculated as

$$\begin{aligned} 34.9 \times 0.0185 \\ = 0.645\,65. \end{aligned}$$

This figure is rounded to 0.646 before the calculation is completed. So finally,

$$\begin{aligned} (34.87 \times 0.018\,54) - 8.575 \\ \simeq (34.9 \times 0.0185) - 8.58 = 0.645\,65 - 8.58 \\ \simeq 0.646 - 8.58 = -7.934 \\ \simeq -7.93 \text{ (to three significant figures).} \end{aligned}$$

Comparing this with the exact answer of $-7.928\,510\,2$, you can see that the result is still quite accurate, considering that only three significant figures were retained at each stage of the calculation.

Exercise 2

Use three figure arithmetic to evaluate

$$(0.2745 \times 73.841) - 6.1879.$$

(Solution on p. 55)

Again, in this exercise, three figure arithmetic gives us a good approximation to the answer, but this is not always the case, as you will see in the television programme.

Although a computer or calculator works far more accurately than this, the arithmetic used in these machines is analogous to the process described above. So the difficulties which arise using three figure arithmetic will also arise when using computers or calculators.

Now watch the television programme. Concentrate on the methods of solution which are shown, rather than on the arithmetical details. The following subsections all contain reference to the television programme. Specific references will be marked with a  symbol.

3.2 Ill-conditioning

The definition of **ill-conditioning** of a set of linear simultaneous equations is similar to that given in the unit on recurrence relations.

A set of linear simultaneous equations is said to be **ill-conditioned** if small changes in the data lead to large changes in the solution.

The 'data' in the case of simultaneous equations refers either to the coefficients of the left-hand sides or to the right-hand sides of the equations.

Normally, if slight changes are made to the data in a set of simultaneous equations, then the solution only changes slightly, as the following example demonstrates.

Example 2

You can check that the simultaneous equations

$$\begin{aligned} x + y &= 5 \\ x - 2y &= -1 \end{aligned}$$

have the exact solution

$$x = 3, \quad y = 2.$$

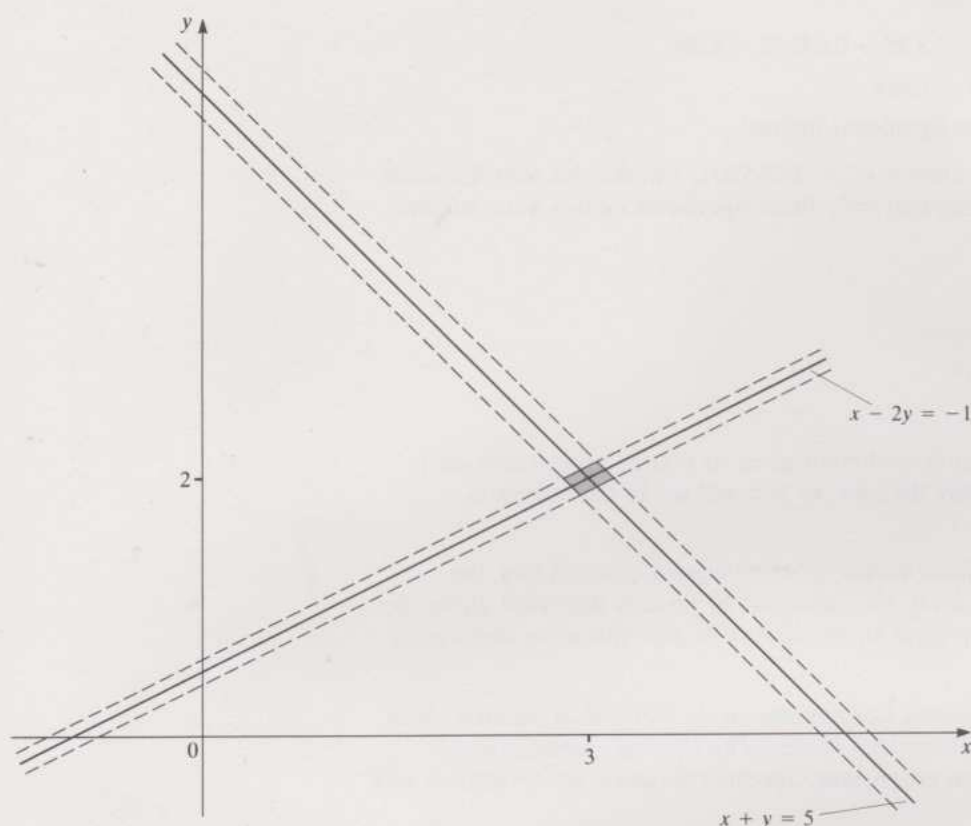
Suppose, now, that slight changes are made to the right-hand sides of these equations, and we solve the equations

$$\begin{aligned} x + y &= 5.02 \\ x - 2y &= -1.01 \end{aligned}$$

These equations have the exact solution

$$x = 3.01, \quad y = 2.01,$$

which as you can see is very close to the original solution. The geometric interpretation of small changes made to the right-hand sides of these equations is shown in the figure below.



The figure above shows the two lines. The dotted lines indicate how much the lines might vary if there were small changes in the right-hand sides of the equations, and the shaded area indicates the region in which we would expect the solution of our equations to lie if these small changes were made. As you can see in this case, the solution we would get to an inexact problem would be a reasonable approximation to the solution of the accurate problem, as all the points in the shaded region are reasonably close to $x = 3, y = 2$ —the exact solution.

Usually, when small changes are made in the data, the solution of the resulting problem is very close to that of the original problem. When solving problems we depend on this fact as in any practical context there would almost certainly be small errors in the coefficients of sets of equations, and yet we have to rely on the solution of this inexact problem being close to the accurate one.

However, this is not always the case, as the following example will demonstrate.

Example 3

You can check that the simultaneous equations

$$\begin{array}{ll} x - 2y = -1 & E_1 \\ 10x - 19y = -8 & E_2 \end{array}$$

have the exact solution.

$$x = 3, \quad y = 2.$$

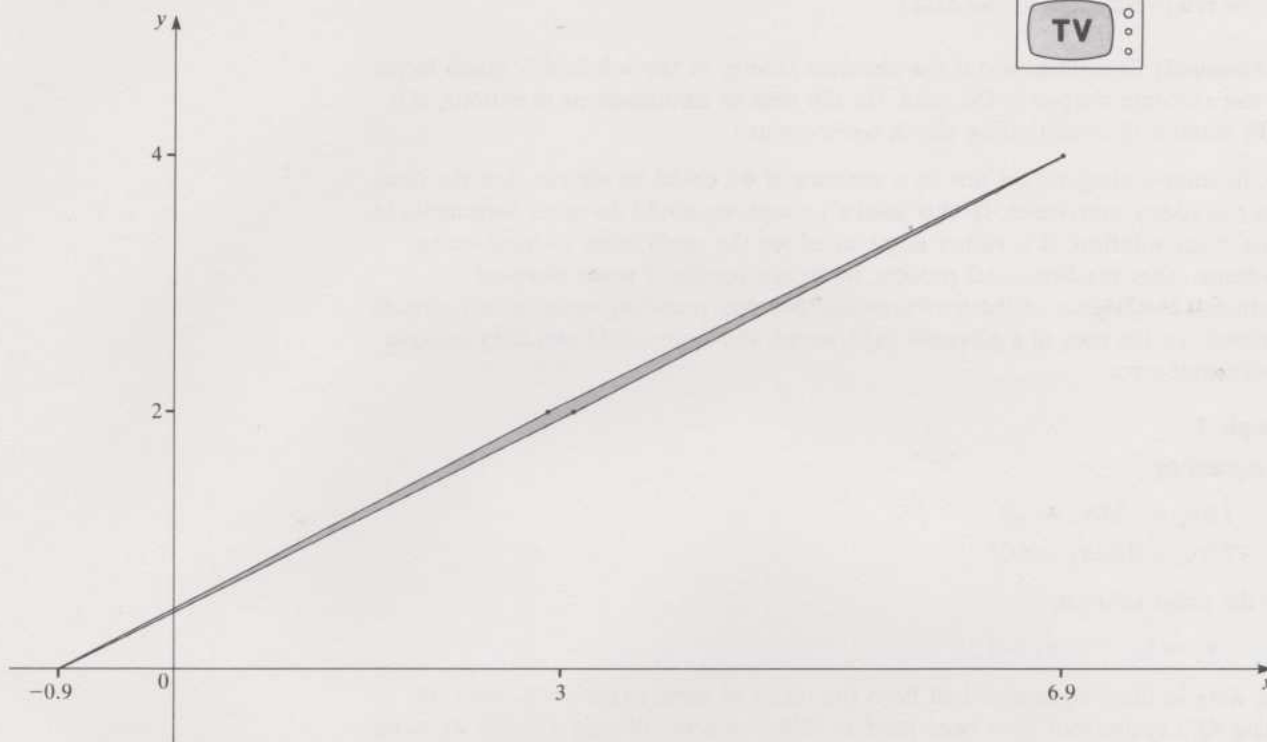
Once again, we shall make small changes to the right-hand sides of the equations. Solving the slightly changed equations

$$\begin{array}{ll} x - 2y = -1.1 & E_3 \\ 10x - 19y = -8 & E_2 \end{array}$$

we find that the exact solution to this slightly altered set is

$$x = 4.9, \quad y = 3.$$

This time, the solution of the slightly altered set of equations bears no resemblance to the solution of the original equations, and we see that the equations are ill-conditioned. The geometric interpretation of this situation can be seen in the figure below.



As the equations are represented geometrically by two nearly parallel lines, we have not drawn in all the lines corresponding to those in the figure for Example 2, because this would make the diagram confusing. This figure simply illustrates the remarkably elongated region in which you could expect the solution to lie if small inaccuracies were present in the right-hand sides of these particular equations. This shows that the solution of these slightly inaccurate equations could in fact be a long way from the accurate solution $x = 3, y = 2$. In other words, with this particular pair of equations, a small variation in the data could produce a large change in the solution.

It is obviously very important to know whether a set of simultaneous equations is ill-conditioned. If it is, the solution obtained should not be considered as reliable.

The geometry shows us that if the lines representing two simultaneous equations are nearly parallel, then the equations will be ill-conditioned. Whereas this concept cannot really be extended to larger sets of equations, the equivalent algebraic concept can. The gradients of the two lines are obtained from the left-hand side coefficients of the equations. If we write these coefficients in matrix form

$$A = \begin{bmatrix} 1 & -2 \\ 10 & -19 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

then an equivalent algebraic way of saying that the lines are nearly parallel would be to note that R_2 is nearly a multiple of R_1 . (R_2 is roughly nine or ten times as big as R_1 .) In other words, the rows of A are nearly linearly dependent.

In general, if a set of linear simultaneous equations is represented by a matrix $A|b$, then if the rows of A are 'nearly' linearly dependent, ill-conditioning is likely to occur.

As you saw in Section 2, linear dependence will show up as a row of zeros if Gaussian elimination is performed on a matrix. So if the rows were nearly linearly dependent we would expect a row in the elimination to be 'nearly' zero.

Perhaps we should mention at this stage that, as in the case of recurrence relations in *Unit 1*, there are two kinds of ill-conditioning.

Suppose that a change is made in the data of a set of simultaneous equations. Then these equations are:

- (i) **relatively** ill-conditioned if the relative change in the solution is much larger than the relative change in the data;
- (ii) **absolutely** ill-conditioned if the absolute change in the solution is much larger than the absolute change in the data. (In the case of simultaneous equations, it is usually relative ill-conditioning which concerns us.)

Now, ill-conditioning would not be a problem if we could be certain that the data in the equations were exact. In this unlikely event, we could do exact arithmetic to get the exact solution. It is rather more usual for the coefficients to have come from some other mathematical process, or be the results of some physical experiment. In the case of the mathematical process, rounding error would already be present. In the case of a physical experiment, the data would certainly contain experimental error.

Example 4

The equations

$$\begin{aligned}1.0x_1 + 2.0x_2 &= 3.0 \\ 2.0x_1 + 4.02x_2 &= 6.02\end{aligned}$$

have the exact solution

$$x_1 = 1, \quad x_2 = 1.$$

If the data in these equations had been the result of some experiment, then the reading 4.02 might well have been read as 3.98—an error of only 1%. So we could just as easily have landed up with the equations

$$\begin{aligned}1.0x_1 + 2.0x_2 &= 3.0 \\ 2.0x_1 + 3.98x_2 &= 6.02\end{aligned}$$

which have the exact solution

$$x_1 = 5, \quad x_2 = -1.$$

Quite a different solution!

It should be noted that the large change in the solution cannot be attributed either to the method used to solve the problem, or to the arithmetic, as the answers given are the exact solutions of the problems. The general problem of detecting ill-conditioning is difficult. Sometimes, as in the bar problem on television, you get results which are clearly impossible. The price of Guinness couldn't possibly be -18p and thus you are alerted to the fact that something is wrong. However, that was lucky! *The direct approach to detecting ill-conditioning is to vary the data slightly, and see if the solution of the altered equations is very different from the original one.* This would be very tedious without the use of a computer. You will be given the opportunity to do this using the computer package in the next section.

The cure for ill-conditioning is fairly drastic. We can abandon the equations in hand, and try to find some more data—or even abandon the whole model. An attempt to find new data was made in the bar problem by ordering another couple of rounds of drinks ...

3.3 The problem of rounding errors

As you saw in Subsection 3.1, a calculator or computer introduces error into a calculation. To start with, these machines have to work with approximations to numbers like $\frac{1}{3}$ or $\sqrt{2}$, and then as the calculation (Gaussian elimination, for

example) progresses, at each step rounding takes place. After a while, most of the numbers in the calculation will have *rounding error* in them. It becomes a matter of prime concern to try and prevent this error (and any other error present) from building up to the point that the answers obtained bear little or no resemblance to the correct solutions.

In Subsection 3.1, the simple example and exercise using three figure arithmetic produced an answer which was a good approximation to the exact answer. Unfortunately, there are circumstances in which this is not the case.

Example 5

It can be seen that the exact value of $\frac{(\sqrt{2})^2 - 1.98}{0.01}$ is 2, whereas using three-figure arithmetic,

$$\begin{aligned}\frac{(\sqrt{2})^2 - 1.98}{0.01} &\simeq \frac{(1.41)^2 - 1.98}{0.01} \\ &\simeq \frac{1.99 - 1.98}{0.01} \\ &= 1.\end{aligned}$$

Thus a very small error made in measuring $\sqrt{2}$ (0.3%) has produced a 50% change in the answer.

The story of error and error build-up is an extremely complicated one. In this subsection, we look at just one particularly bad situation which can arise when solving simultaneous equations.

Example 6

The television programme discusses the solution, using three figure arithmetic, of the problem

$$\begin{aligned}x - \sqrt{2}y + z &= 1 \\ -\sqrt{2}x + 2y &= 1 \\ 2y - \sqrt{2}z &= 1\end{aligned}$$

Using exact arithmetic, this produces a solution

$$\begin{aligned}x = y = z &= 1 + \frac{\sqrt{2}}{2} \\ &= 1.71 \text{ (to three significant figures).}\end{aligned}$$

The only rounding done in this process is in the final form of the answer.

Now, using three-figure arithmetic to do this same problem, the matrix representing these equations is

$$\left[\begin{array}{ccc|c} 1.00 & -1.41 & 1.00 & 1.00 \\ -1.41 & 2.00 & 0 & 1.00 \\ 0 & 2.00 & -1.41 & 1.00 \end{array} \right] \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{matrix}$$

The elimination goes as follows:

Stage 1(a)

$$\begin{array}{c} -1.41 \\ 0 \end{array} \left[\begin{array}{ccc|c} 1.00 & -1.41 & 1.00 & 1.00 \\ 0 & 0.01 & 1.41 & 2.41 \\ 0 & 2.00 & -1.41 & 1.00 \end{array} \right] \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{matrix}$$

Stage 1(b)

$$\begin{array}{c} 200 \end{array} \left[\begin{array}{ccc|c} 1.00 & -1.41 & 1.00 & 1.00 \\ 0 & 0.01 & 1.41 & 2.41 \\ 0 & 0 & -283 & -481 \end{array} \right] \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{matrix}$$

Note that just a record of the multipliers has been kept at each stage of the elimination. For example, in Stage 1(a), where before we would have written the comment ' $R_2 + 1.41R_1$ ' on the left of the matrix opposite R_{2a} , we now just record the number -1.41 , being that *multiple* of R_1 which is *subtracted* from R_2 to produce R_{2a} . We shall stick to this notation for the rest of this unit.

Doing the back substitution:

R_{3b} gives

$$-283z = -481,$$

hence

$$z = 1.70 \text{ (not bad).}$$

R_{2a} gives

$$0.01y + 1.41z = 2.41,$$

hence

$$\begin{aligned} y &= \frac{2.41 - (1.41 \times 1.70)}{0.01} \\ &= 1.00 \text{ (very bad);} \end{aligned}$$

and finally R_1 gives

$$1.00x - 1.41y + 1.00z = 1.00$$

hence

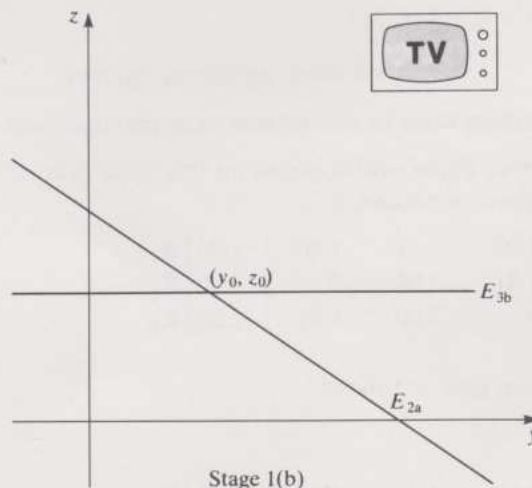
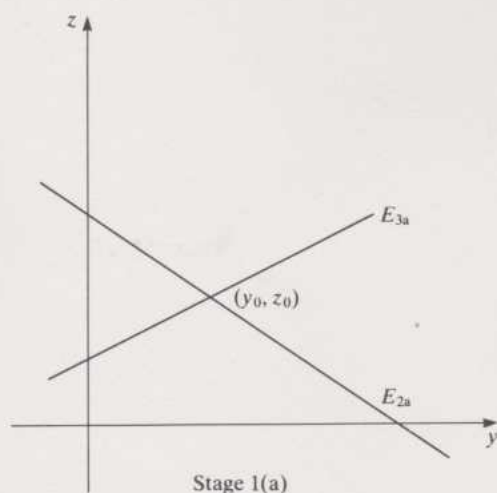
$$\begin{aligned} x &= 1.00 + (1.41 \times 1.00) - (1.00 \times 1.70) \\ &= 0.710 \text{ (hopeless).} \end{aligned}$$

The solution

$$x = 0.710, \quad y = 1.00, \quad z = 1.70$$

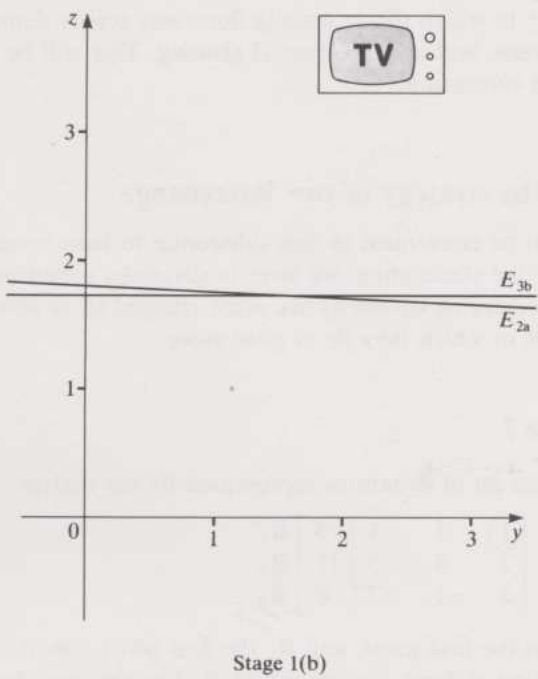
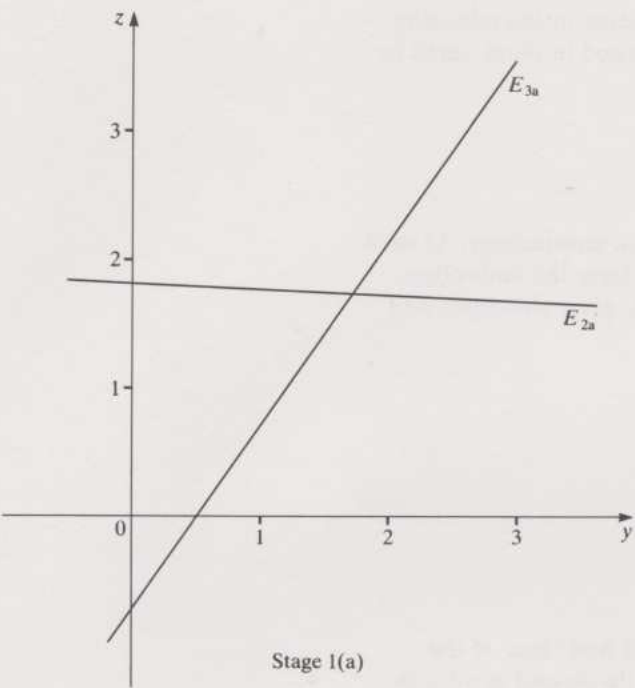
is nowhere near the solution of the exact problem. As this is a disastrous state of affairs, it is worth analysing in more detail.

To help understand what went wrong in this particular problem, let us first look at the graphical interpretation of Stages 1(a) and 1(b) of the elimination process, for a more typical case in which nothing goes wrong.



When Stage 1(b) of the elimination is done, the line representing the equation E_{3a} is replaced by a line parallel to the y -axis. The lines representing E_{2a} and E_{3a} have the same point of intersection as those representing E_{2a} and E_{3b} (see the diagram above). If the arithmetic were exact the solution of these two pairs of equations would be exactly the same.

Now let us look at what happened in Example 6.



In Stage 1(a) in Example 6, we had the equations

$$\begin{aligned} 0.01y + 1.41z &= 2.41 & E_{2a} \\ 2.00y - 1.41z &= 1.00 & E_{3a} \end{aligned}$$

From the diagram above, it can be seen that the trouble arises because E_{2a} is practically parallel to the y -axis. Thus, although the equations in Stage 1(a) are not ill-conditioned, the ones we obtain in Stage 1(b) are, as E_{3b} is always parallel to the y -axis. Now, inaccuracy has been brought into the calculation from:

- (i) approximation of $\sqrt{2}$ as 1.41,
- (ii) the use of three-figure arithmetic;

so we would hardly expect the resulting solution to bear any resemblance to the solution of the accurate problem.

Whereas ill-conditioning was a property of the equations we started with, this situation is rather different. Here, the ill-conditioning is induced by the method we used to solve the equations. We began with a problem which was not ill-conditioned to start with, and yet the method induced ill-conditioning. This induced ill-conditioning is often referred to as **induced instability**.

In order to cope with larger sets of equations, we need to look for an algebraic as well as a geometric explanation.

One way of seeing what went wrong between Stage 1(a) and Stage 1(b) is to look at how R_{3b} is derived.

$$R_{3a} - 200 R_{2a} \quad \left[\begin{array}{ccc|c} 1.00 & -1.41 & 1.00 & 1.00 \\ 0 & 0.01 & 1.41 & 2.41 \\ 0 & 0 & -283 & -481 \end{array} \right] \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

We know that

$$R_{3b} = R_{3a} - 200R_{2a}.$$

Now $200R_{2a}$ is likely to swamp any information held in R_{3a} because of the large size of the multiplier, so

$$R_{3b} \simeq -200R_{2a}.$$

The large multiplier has ensured that R_{2a} and R_{3b} are nearly linearly dependent, and hence are likely to be ill-conditioned.

The argument above suggests that if we want to avoid inducing instability, we must try to adapt the method of Gaussian elimination to avoid large multipliers. The way in which this is usually done was briefly demonstrated in the television programme, and is called **partial pivoting**. This will be discussed in more detail in the next subsection.

3.4 The strategy of row interchange

It would be convenient in this subsection to have some extra terminology. At each stage of the elimination, we have to divide by a number to form the multipliers. The numbers we divide by are often referred to as **pivots** (or **pivot elements**), and the rows in which they lie as **pivot rows**.

Example 7

Given the set of equations represented by the matrix

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 2 & 4 & 5 & 11 \\ 3 & -1 & -2 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

then 1 is the first pivot, and $\mathbf{R}_1$ the first pivot row. After the first stage of the elimination is done, the second pivot, 2 in this case, lies in the second pivot row $\mathbf{R}_{2a}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & \textcircled{2} & 3 & 5 \\ 0 & -4 & -5 & -9 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

The subsequent pivots can be found after further stages in the elimination.

(a) Essential row interchange

In Subsection 3.3, we saw that it was advisable to avoid large multipliers. Now, as large multipliers are caused by small pivots, we shall have to avoid small pivots. To start with, let us look at what happens when we meet a *zero* pivot. When we meet a zero pivot, the idea of altering the order of the equations (or the equivalent rows in a matrix) is fairly instinctive, as the following example demonstrates.

Example 8

Given the set of equations

$$\begin{array}{rcl} x_2 + x_3 = 2 & E_1 \\ 2x_1 + 4x_2 - x_3 = 5 & E_2 \\ x_1 + 2x_2 - x_3 = 2 & E_3 \end{array}$$

represented by the matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & 4 & -1 & 5 \\ 1 & 2 & -1 & 2 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

it would be impossible to do Gaussian elimination on the matrix as it stands. The reason for this is that the first multiplier would be $\frac{2}{0}$, which is of no use. These equations have a solution $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and, as you will see later, this is a unique solution. So the elimination process as described so far would not cope with this situation.

The trouble arose because there was a zero pivot. In other words, the equation $x_2 + x_3 = 2$ is not at all suitable to have as the first equation. The obvious remedy is to change the order of the equations. This is equivalent to interchanging two rows of the matrix $\mathbf{A}|\mathbf{b}$. If we were doing the problem above, we could interchange

two rows to obtain the matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & -1 & 2 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

We could then proceed with the elimination to obtain

$$\frac{1}{2} \left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

and hence obtain a unique solution.

Where a row interchange has taken place, a double-headed arrow will be used to demonstrate which rows have been interchanged, but the rows will keep their sequential numbering.

To be able to use Gaussian elimination on a set of equations, we must check at each stage that the appropriate pivot is not zero. If a zero pivot is encountered, an **essential row interchange** has to be made, such as the one we have just seen.

The instruction of which row to interchange with which has got to be general enough to cover all possibilities. For instance, exchanging $\mathbf{R}_1$ and $\mathbf{R}_2$ would not work in the case of the equations

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_2 + 2x_3 &= 3 \\ -2x_1 - x_2 + 5x_3 &= 2 \end{aligned}$$

as once again, we would get a zero pivot.

The general rule we adopt for essential row interchange is as follows:

If any pivot element is zero, interchange this pivot row with the row containing the coefficient of highest modulus in the column underneath the pivot.

Then, short of the very unlikely possibility of the whole column being zero, this will produce a non-zero pivot.

Note that this process of essential row interchange can be done at any stage of the elimination, as you can see in the next exercise.

Exercise 3

Use Gaussian elimination with essential row interchanges to solve the equations

$$\begin{array}{rcl} x_1 - x_2 + x_3 - x_4 & = & 4 \\ 2x_1 - 2x_2 + 3x_3 + 2x_4 & = & 5 \\ x_2 - 2x_3 - 3x_4 & = & 0 \\ -x_2 + 4x_3 + x_4 & = & 4 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}$$

(Solution on p. 55)

Example 9

To investigate the type of solution of the equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 + 2x_3 &= 4 \\ x_1 + x_2 + 3x_3 &= q \end{aligned}$$

for various values of q , once again we use the elimination process.

Stage 1(a) gives us

$$\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & q-3 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

In a rare event such as this, it is impossible to find a non-zero pivot, and the elimination process as described has broken down. At this stage, $\mathbf{R}_{2a}$ gives us

$$x_3 = 1.$$

On the other hand, $\mathbf{R}_{3a}$ gives us

$$2x_3 = q - 3.$$

So, if these equations are consistent, then $q = 5$. If q has any other value, then the equations will be inconsistent. The reason that the possibility of inconsistency did not show up in the elimination was that we were not able to complete the process.

Finally, it is instructive to look at what would have happened had exact arithmetic been performed on the second television example (i.e. Example 6 on p. 33). The matrix representing the equations is

$$\left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 1 \\ -\sqrt{2} & 2 & 0 & 1 \\ 0 & 2 & -\sqrt{2} & 1 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

The first stage of the elimination would give us

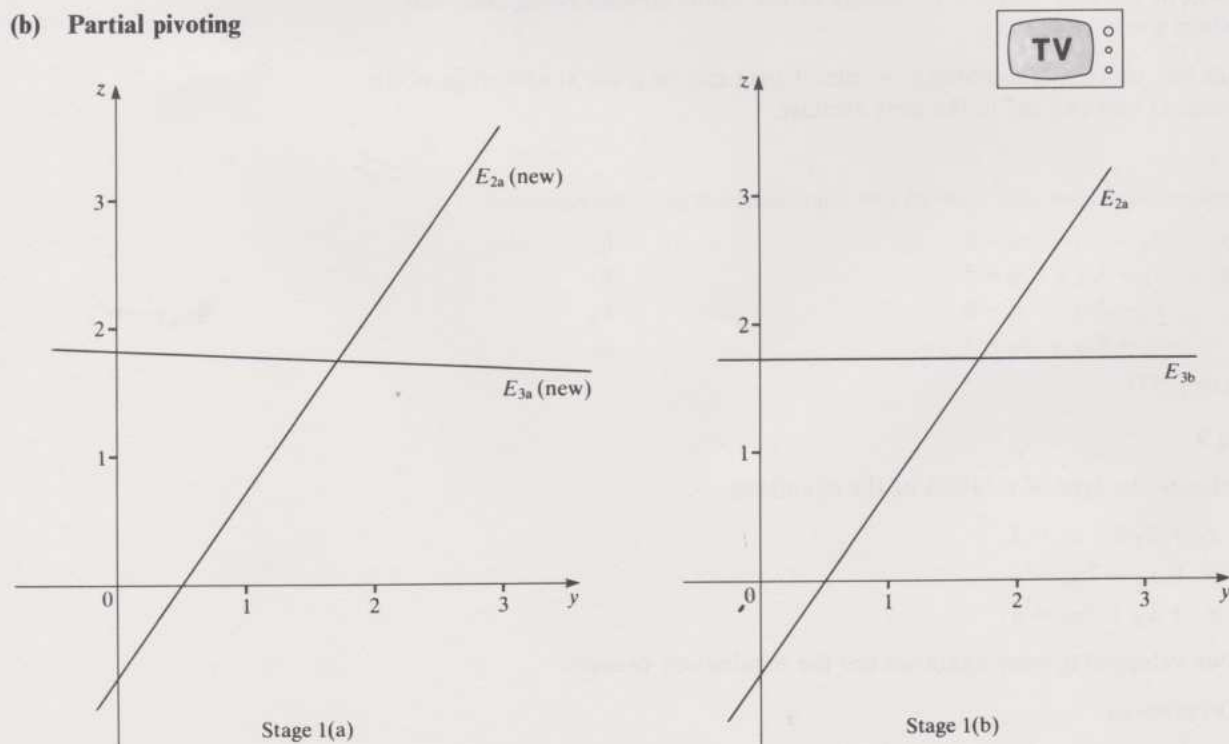
$$\begin{array}{c} -\sqrt{2} \\ 0 \end{array} \left[\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 1 \\ 0 & 0 & \sqrt{2} & 1 + \sqrt{2} \\ 0 & 2 & -\sqrt{2} & 1 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

In this case, we would now have to make an essential row interchange to be able to continue the process. The equivalent stage in the inexact arithmetic was

$$\begin{array}{c} -1.41 \\ 0 \end{array} \left[\begin{array}{ccc|c} 1 & -1.41 & 1 & 1 \\ 0 & 0.01 & 1.41 & 2.41 \\ 0 & 2 & -1.41 & 1.00 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

With hindsight, perhaps we should have realized that this small pivot element was very dangerous, and we should have changed the rows over anyway—even though the pivot wasn't zero. Certainly the existence of this small pivot (0.01) caused the large multiplier (200)—which in turn induced instability into a problem where there was none to start with.

(b) Partial pivoting



In the second television example (Example 6), the first stage of the elimination gives us the equations

$$0.01y + 1.41z = 2.41$$
$$2.00y - 1.41z = 1.00$$

$$E_{2a}$$
$$E_{3a}$$

Had we interchanged the order of these equations (see figure on p. 38)

$$2.00y - 1.41z = 1.00$$
$$0.01y + 1.41z = 2.41$$

$$E_{2a} \text{ (new)}$$
$$E_{3a} \text{ (new)}$$

and performed the second stage of the elimination on them in this new order, then the problem of ill-conditioning would not have arisen, as you can see from the diagram. Thus, the solution we would get using the equations in this new order would be reasonably accurate. In an attempt to avoid inducing ill-conditioning into a problem we employ a strategy called **partial pivoting**. This is the same as the procedure described for essential row interchange, except that we interchange the rows (if necessary) automatically at *each* stage of the elimination. As before, at the beginning of each stage of the elimination, we choose to interchange two rows of the matrix, so that the pivot has the highest possible modulus. This should ensure that the multipliers are all less than or equal to one in modulus.

Partial pivoting is the process of automatically altering the order of the equations at the beginning of each stage of the elimination in the following way.

Interchange the current pivot row with the row containing the coefficient of highest modulus in the column underneath the pivot.

Example 10

We shall solve the equations from the television programme again:

$$1.00x - 1.41y + 1.00z = 1.00$$
$$-1.41x + 2.00y = 1.00$$
$$2.00y - 1.41z = 1.00$$

$$E_1$$
$$E_2$$
$$E_3$$

This time, however, we shall use partial pivoting as well as three-figure arithmetic.

The matrix representing the equations is

$$\left[\begin{array}{ccc|c} 1.00 & -1.41 & 1.00 & 1.00 \\ -1.41 & 2.00 & 0 & 1.00 \\ 0 & 2.00 & -1.41 & 1.00 \end{array} \right]$$

$$\mathbf{R}_1$$
$$\mathbf{R}_2$$
$$\mathbf{R}_3$$

Before performing the first stage of the elimination, interchange $\mathbf{R}_1$ and $\mathbf{R}_2$, as -1.41 in $\mathbf{R}_2$ is the coefficient with the highest modulus in the first column. This gives us

$$\left[\begin{array}{ccc|c} -1.41 & 2.00 & 0 & 1.00 \\ 1.00 & -1.41 & 1.00 & 1.00 \\ 0 & 2.00 & -1.41 & 1.00 \end{array} \right]$$

$$\mathbf{R}_1 \leftarrow$$
$$\mathbf{R}_2 \leftarrow$$
$$\mathbf{R}_3$$

and then we do the first stage of the elimination:

$$-0.709 \left[\begin{array}{ccc|c} -1.41 & 2.00 & 0 & 1.00 \\ 0 & 0.01 & 1.00 & 1.71 \\ 0 & 0 & 2.00 & -1.41 \end{array} \right]$$

$$\mathbf{R}_1$$
$$\mathbf{R}_{2a}$$
$$\mathbf{R}_{3a}$$

Before the second stage of the elimination is done, we interchange $\mathbf{R}_{2a}$ and $\mathbf{R}_{3a}$ to obtain the coefficient with the largest modulus for the second pivot. This gives us

$$\left[\begin{array}{ccc|c} -1.41 & 2.00 & 0 & 1.00 \\ 0 & 2.00 & -1.41 & 1.00 \\ 0 & 0.01 & 1.00 & 1.71 \end{array} \right]$$

$$\mathbf{R}_1$$
$$\mathbf{R}_{2a} \leftarrow$$
$$\mathbf{R}_{3a} \leftarrow$$

and hence the second stage of the elimination:

$$0.005 \left[\begin{array}{ccc|c} 1.41 & 2.00 & 0 & 1.00 \\ 0 & 2.00 & -1.41 & 1.00 \\ 0 & 0 & 1.01 & 1.71 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

The back substitution gives us

$$1.01z = 1.71.$$

Hence

$$z = 1.69 \text{ (to three significant figures).}$$

From $\mathbf{R}_{2a}$, we get

$$2y - 1.41z = 1.00.$$

Hence

$$\begin{aligned} y &= \frac{1.00 + (1.41 \times 1.69)}{2} \\ &= 1.69 \text{ (to three significant figures).} \end{aligned}$$

Finally, $\mathbf{R}_1$ gives us

$$-1.41x + 2y = 1.$$

Hence

$$\begin{aligned} x &= \frac{1 - (2 \times 1.69)}{-1.41} \\ &= 1.69 \text{ (to three significant figures).} \end{aligned}$$

This solution $x = y = z = 1.69$ is a far better estimate than that obtained without partial pivoting.

Finally, note that partial pivoting is not an absolute safeguard against induced instability. It does have the advantage of being easy to implement and is a good 'rule of thumb'. If, in addition, we try to ensure that the coefficients of the equations we start with are roughly of the same size, then partial pivoting usually works. This can often be done by multiplying the original equations by a suitable factor. This is called **scaling** the equations. Equations which arise in practice are very often naturally scaled.

The process of partial pivoting used with Gaussian elimination ensures that the largest possible value is used as a pivot at each stage of the elimination. This usually helps to control the build-up of error during the process.

Summary of Section 3

1. The section introduces some new terminology.

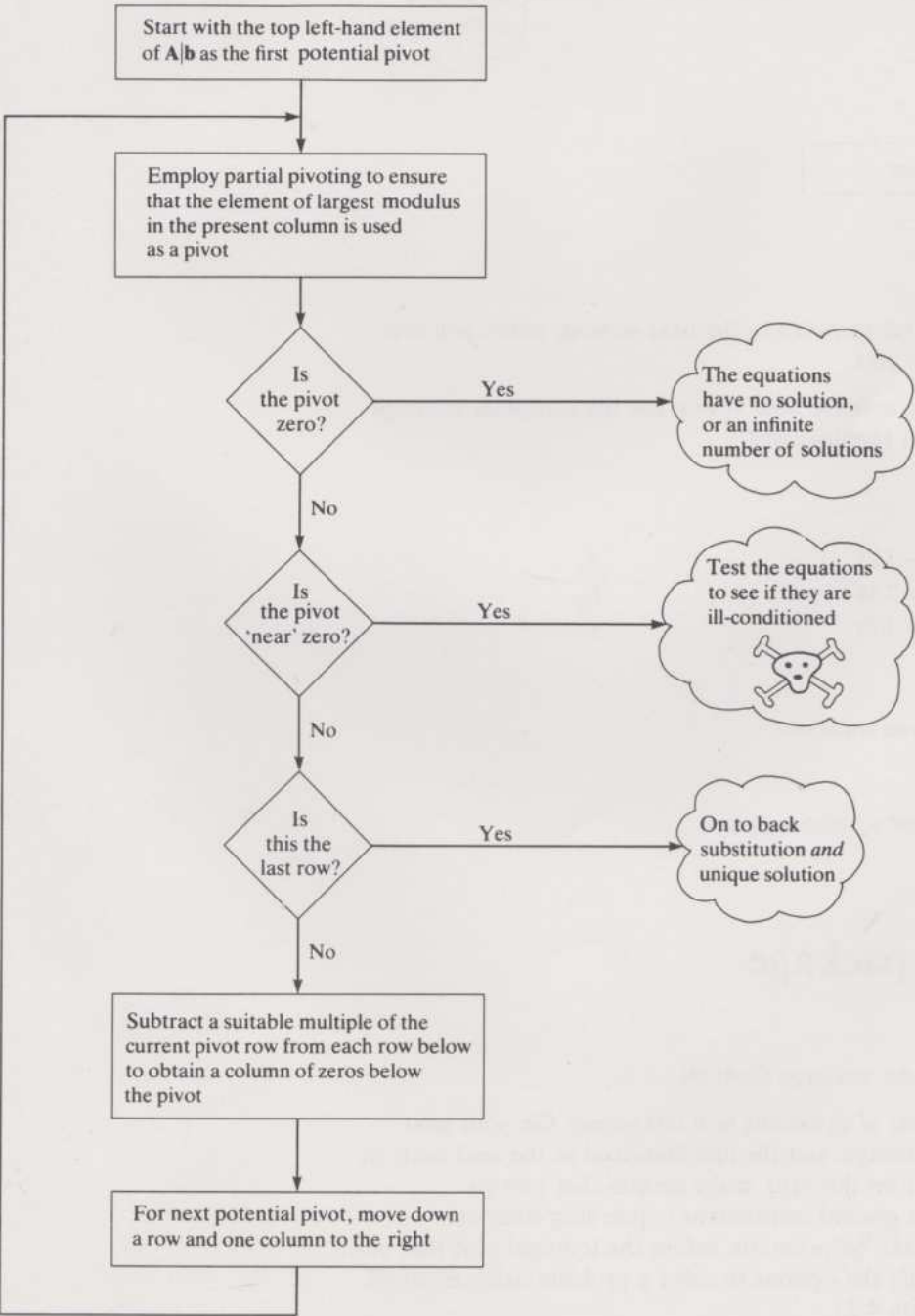
- (i) The k th **pivot** (or **pivot element**) is the one by which we divide to obtain the multiplier in the k th step of the elimination.
- (ii) The **pivot row** is the row in which the current pivot lies.
- (iii) **Essential row interchange** is the process of interchanging the rows of a matrix when the potential pivot is zero. This can be done by interchanging the current pivot row with whichever of the rows beneath it will provide the pivot with the largest modulus.
- (iv) **Partial pivoting** is the process described above, with the difference that this is done at *each* stage of the elimination.

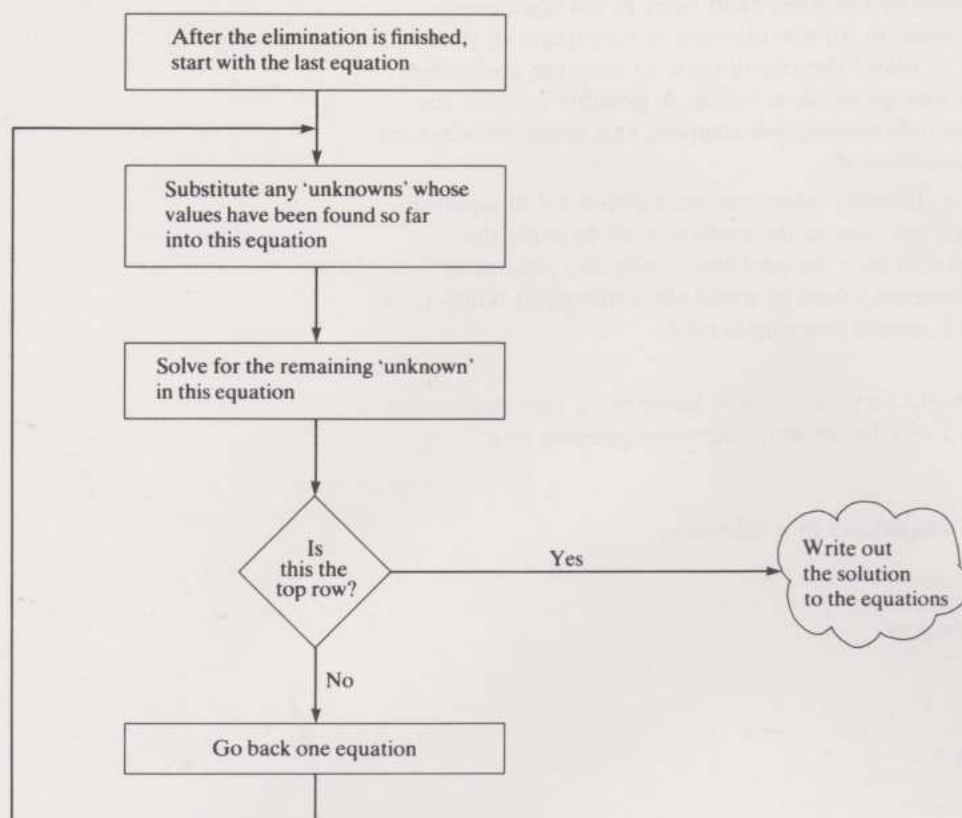
2. The section shows some of the difficulties presented by simultaneous equations and their solution.

- (i) **Ill-conditioning.** A set of equations is said to be **ill-conditioned** if a ‘small’ change in the data (the coefficients, or the right-hand sides of the equations) produces ‘large’ changes in the solution. Ill-conditioning is a property of the equations themselves. One way to detect the condition is to vary the coefficients slightly and see if there is much change in the solution. A possible cure for the situation is to obtain some more independent information, and hence obtain a set of equations which are not ill-conditioned.
- (ii) **Induced instability.** This is a difficulty where we start with a set of equations which are not ill-conditioned and yet, due to the method used to solve the equations, ill-conditioning is induced into the problem during the process of solution. To try to avoid this happening (and to avoid the subsequent build-up of error), Gaussian elimination with partial pivoting is used.

3. The following flow charts might serve as a useful guide as to how to proceed when systematically solving a set of n linear simultaneous equations in n unknowns.

(i) The elimination process for n equations in n unknowns



(ii) The back substitution process for n equations in n unknowns**End of section exercise**

The exercises on ill-conditioning are included in the next section, where you can use the computer package to help you.

If you have time, do the following exercise now. If not, use the computer package to solve the problem on your next terminal visit.

Exercise 4

Check that the equations

$$1.62x_1 + 1.10x_2 + 0.65x_3 = 3.37 \quad E_1$$

$$6.18x_1 + 4.20x_2 - 3.04x_3 = 7.34 \quad E_2$$

$$4.65x_1 - 3.05x_2 + 2.10x_3 = 3.70 \quad E_3$$

have the exact solution

$$x_1 = x_2 = x_3 = 1.$$

Use Gaussian elimination to solve these equations:

- (i) without partial pivoting,
- (ii) with partial pivoting.

Do each stage of the arithmetic to four significant figures.

(Solution on p. 56)

4 The computer package

4.1 Introduction

This section describes the computer package SIMLIN.

SIMLIN is designed to solve n sets of equations in n unknowns. On your next terminal visit you will use this package, and the one described in the next unit. As you will have a lot of work to do on this visit, make certain that you go adequately prepared. Re-read the general instructions concerning computer packages given in *Unit 1*, and make quite certain before the terminal visit that you know the order in which to specify the options to solve a problem using SIMLIN. (An example is given in Subsection 4.2.)

4.2 A worked example

Suppose we wish to use the computer package SIMLIN to solve the equations

$$\begin{array}{rcl} 2x_1 - x_2 - 2x_3 & = & 4 \\ x_1 - 2x_2 + x_3 & = & -4 \\ x_1 + 5x_2 + x_3 & = & 13 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

To do this, only a subset of the options described in Subsection 4.3 is required. The following computer dialogue illustrates the options needed for this particular problem. The information you have to type in is underlined. The rest of the printing is supplied by the computer.

When you have logged on, and obtained the computer package SIMLIN, the computer will ask you to specify an option. You could then proceed as follows:

```
OPTION? 10
HOW MANY EQUATIONS DO YOU WANT TO SOLVE? 3
OPTION? 11
TYPE IN THE MATRIX A, ONE ROW AT A TIME, SEPARATING THE
ELEMENTS BY COMMAS.
ROW 1? 2, -1, -2
ROW 2? 1, -2, 1
ROW 3? 1, 5, 1
OPTION? 12
TYPE IN THE ELEMENTS OF MATRIX B, SEPARATING THE
ELEMENTS BY COMMAS.
ELEMENTS? 4, -4, 13
OPTION? 20
GAUSSIAN ELIMINATION METHOD WITH PARTIAL PIVOTING
OPTION? 42
PRINT SOLUTION ONLY
OPTION? SOLVE
SOLUTION
VARIABLE VALUE
    1    2.03571
    2    2.42857
    3   -1.17857
OPTION?
```

At this stage, you are free to run another problem. If you type STOP at this stage, you will be able to call another computer package, or to end your session.

It is worth noting that the instruction to solve the problem (SOLVE) should only be given *after* all the data and the method of solution have been specified. If you ask for the problem to be solved before this stage, an error message will be printed, asking you for more information.

In the following description, the options we used in the example above are marked with an asterisk.

4.3 A list of options in SIMLIN

The options in this package are described under the same general headings as those in RECREL (*Unit 2*).

Command options

The control options are essentially the same as those in RECREL. Unlike the rest of the options described in this section, the command options have names rather than numbers associated with them.

- OPTIONS —this option prints the list of options available to you in this package.
- SOLVE* —this option computes the solution. It is the last option you use when solving a problem. Before this is used, all the data and the method of solution must have been given to the computer. If any

of this information is missing, the computer will print an error message.

- HELP** —help may be obtained if you need it, by typing **HELP** at any stage when a question mark appears. If you type **HELP** after the computer has printed **OPTION?**, you will be asked questions to help you to decide what option to use next. These questions require an answer **YES** or **NO**. If you type **HELP** while using an option, you will be given help as to how to use this option properly.
- LIST** —this option prints out the information held in the computer about the current problem. It is advisable to ask for **LIST** before you ask for the problem to be solved. In this way, you can check that the computer is holding the correct information.
- STOP** —this option is your means of exit from the package. You would only use this option if
- (i) you wanted to use another package, or
 - (ii) you wanted to log-off the computer.
- ANSWER** —this gives part answers to the end of section exercises so that you can check that you are on the right lines. To the question

EXERCISE?

you respond with the number of the exercise you are interested in.

The rest of the options in this package are obtained by typing a number when the computer types **OPTION?**

Problem options

10* Specify the number of equations

You will use option 10 to specify how many equations you want to solve. (You are limited to a maximum of fifteen equations.) This option must be used before you try to give the computer the data of the problem (options 11 and 12).

11* Enter matrix A

You will use option 11 to put the matrix **A** (the left-hand sides of the equations) into the computer, after you have specified the number of equations you want to solve by using option 10. This is done one row at a time. When the message

ROW 1?

appears, type in all the elements in the first row, separating the elements by commas. If you type too few elements an error message and an instruction to re-type the whole row will be printed. If you type too many elements, you will be given a warning message.

12* Enter matrix b

You will use option 12 to put the right-hand sides of the equations into the computer. When the message

ELEMENTS?

appears, type in all the right-hand side elements on the same line, separating them by commas. The error and warning messages are the same as in option 11.

13 Edit A

You can use option 13 if you wish to change a few individual elements in **A**. The two obvious circumstances in which you would want to do this are

- (i) if you made an error in typing in the elements, and
- (ii) if you were testing a problem for ill-conditioning.

Suppose you wanted to change the element in the second row and third column of **A** to 8, and no other change. The computer dialogue would go as follows:

OPTION? 13

CHANGE ELEMENT IN ROW? 2

AND COLUMN? 3

TO HAVE THE VALUE? 8
DO YOU WANT TO CHANGE ANOTHER ELEMENT IN A? NO
OPTION?

Had you answered YES to the last question, you would have been asked to specify the next element you wanted to change.

14 Edit b

You can use option 14 if you wish to change any of the elements of **b** (the right-hand sides of the equations). The reasons you would wish to do this, and the method for doing it, are similar to option 13. Suppose, for instance, you wished to change the right-hand side of the third equation to -7.1 . Then the dialogue with the computer would go as follows:

OPTION? 14
CHANGE ELEMENT IN ROW? 3
TO HAVE THE VALUE? -7.1
DO YOU WANT TO CHANGE ANOTHER ELEMENT IN B? NO
OPTION?

15 Choose a standard problem

To save you having to type in some of the larger problems at the end of this section, and in your tutor-marked assignment, there is a set of problems already stored in the computer. Each stored problem has a name. For instance, Exercise 4 at the end of this section is called PROB1. The following dialogue with the computer will access this problem.

OPTION? 15
PROBLEM NAME? PROB1
OPTION? LIST

It is advisable to print out the standard problem (using LIST) to check that the data is what you expected.

16 Specify number of significant figures

You can use this option if you want each stage of the calculation done to a fixed number of significant figures. For instance, the dialogue

OPTION? 16
HOW MANY SIGNIFICANT FIGURES DO YOU WANT
TO USE IN THE CALCULATION? 3

will cause each stage of the calculation to be done to three significant figures.

You can specify any number of significant figures between 1 and 8. If you do not specify how many significant figures you wish to work to, the computer will work to its normal accuracy.

Method options

These options give you the choice of which variation of Gaussian elimination you wish to use.

20* Partial pivoting

If you choose option 20, the Gaussian elimination process will be done with partial pivoting.

21 User row interchange

This option allows you to choose which row you would like to use as the pivot row at each stage of the elimination. This option should be used in conjunction with the full printout option (option 40), otherwise you won't know which row to use as a pivot row.

22 Essential row interchange

If you choose option 22, then Gaussian elimination will be done without pivoting, unless a zero pivot element is encountered. In this case, an essential row interchange will be made.

Print options

These options give you the choice of how much detail you get printed out in the solutions.

40 Full printout

This option gives the solution, a printout of each stage of the elimination, and the multipliers used to obtain it. It is interesting to see, but don't ask for full printout too often—it takes up a lot of time!

41 Outline printout

This option gives the solution and a printout of the final stage of the elimination.

42* Print solution only

This option gives the solution only (no working on the way).

A note of advice

Once you have specified the options necessary to run your problem, those options will stay in the state that you set them, unless you specifically change them. This can be an advantage or a disadvantage. For instance, if you have just solved three equations in three unknowns and you want to solve another set of three simultaneous equations, you would only have to re-specify options 11 and 12 (enter **A** and **b**)—as the computer retains the rest of the information from the last problem. However, if you have just done a problem using three significant figure arithmetic and subsequently wish to do a problem to the full accuracy of the computer, you must remember to re-specify the number of significant figures (using Option 16).

Summary of Section 4

This section describes the options in the computer package SIMLIN. This package solves sets of n simultaneous equations in n unknowns ($n \leq 15$). A standard sequence of options for solving a given system of equations is:

```
10    Specify the number of equations
11    Enter matrix A
12    Enter matrix b
20    Partial pivoting (method)
42    Solution only
LIST  (optional—but informative)
SOLVE
```

End of section exercises

Exercise 1

Use the computer package SIMLIN to solve the set of simultaneous equations

$$\begin{array}{rcl} 3x_1 - 4.1x_2 + 5.2x_3 & = & 4.1 & E_1 \\ -4.1x_1 + 5.2x_2 + 3x_3 & = & 4.1 & E_2 \\ 5.2x_1 + 3x_2 - 4.1x_3 & = & 4.1 & E_3 \end{array}$$

Exercise 2

Use the computer package SIMLIN to solve the set of simultaneous equations whose matrices are

$$A = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 0 & 6 \\ 4 & -1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Exercise 3

If you have not already done the end of section exercise in Section 3, do it now, using the computer package SIMLIN.

Exercise 4

(i) Solve the set of simultaneous equations whose matrices are

$$A = \begin{bmatrix} 1 & 1.5 & 0.75 & 0.6 & 0.3 \\ 0.75 & 1 & 0.6 & 0.5 & 1.5 \\ 0.6 & 0.75 & 0.5 & 0.43 & 1 \\ 0.5 & 0.6 & 0.43 & 0.37 & 0.75 \\ 0.43 & 0.5 & 0.37 & 0.33 & 0.6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

This problem is already stored in the computer, to save you having to type in the coefficients, and is called **PROB1**.

To obtain **PROB1**, use option 15. When the program types

PROBLEM NAME?

you should type

PROB1

(ii) Is the solution you have found sensitive to small changes in the data?
(Use option 13 to edit the data.)

Exercise 5

Use: (i) Gaussian elimination with essential row interchange,
(ii) Gaussian elimination with partial pivoting, to solve the set of simultaneous equations whose matrices are

$$A = \begin{bmatrix} 17 & 10 & 9 & & & & & & & & \\ 10 & 17 & 8 & 7 & & & & & & & \\ 9 & 8 & 17 & 6 & 5 & & & & & & \\ & 7 & 6 & 17 & 4 & 3 & & & & & \\ & & 5 & 4 & 17 & 2 & 1 & & & & \\ & & & 3 & 2 & 17 & 0 & 1 & & & \\ & & & & 1 & 0 & 17 & 2 & 3 & & \\ & & & & & 1 & 2 & 17 & 2 & 4 & \\ & & & & & & 3 & 2 & 17 & 3 & \\ & & & & & & & 4 & 3 & 17 & \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11.4 \\ 17.9 \\ 16.6 \\ 14.2 \\ -11.7 \\ -3.9 \\ -4.8 \\ 5.7 \\ -7.7 \\ 8.9 \end{bmatrix}$$

where the blank entries in **A** represent zeros. To save time, this problem has been stored in SIMLIN as **PROB2** and can be retrieved using Option 15.

Exercise 6

(i) Solve the set of simultaneous equations whose matrices are

$$A = \begin{bmatrix} 0.45 & 0.1 & 0.16 & 0.09 & 0.04 & -0.54 \\ 0.47 & 0.183 & 0 & 0.18 & -0.31 & -0.36 \\ 0.09 & 0.154 & 0.653 & 0.15 & 0.19 & 0.36 \\ 0.5 & 0.3 & 0.2 & 0.3 & -0.2 & 0.2 \\ -0.17 & -0.177 & -0.163 & -0.18 & 0.32 & 0.08 \\ 0.29 & 0.053 & -0.762 & 0.06 & -0.41 & 1.26 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1.032 \\ 0.3656 \\ 1.6286 \\ 0.74 \\ -0.3794 \\ -2.4176 \end{bmatrix}$$

This problem is stored in SIMLIN as **PROB3**.

(ii) Are the equations ill-conditioned?

5 Special cases

5.1 Homogeneous sets of equations

Sometimes the equations we want to solve are of a special form, where the right-hand sides are all zero:

$$a_1x_1 + b_1x_2 + c_1x_3 = 0$$

$$a_2x_1 + b_2x_2 + c_2x_3 = 0$$

$$a_3x_1 + b_3x_2 + c_3x_3 = 0$$

Such a system is said to be **homogeneous**. The principles developed already show how to deal with this type of equation.

By inspection of the equations above, it can be seen that

$$x_1 = x_2 = x_3 = 0$$

is always a solution. An 'all-zero' solution of this kind is often referred to as **the trivial solution**. The question is, do sets of homogeneous equations ever have any solution other than the trivial one? As with the other examples in this unit, the elimination process will answer this question.

Example 1

Solve the simultaneous equations

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 0 \\ x_1 + 2x_2 + 3x_3 & = & 0 \\ 2x_1 + 3x_2 - x_3 & = & 0 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

Proceeding as usual, we get

$$\begin{array}{l} \text{Stage 1(a)} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array} \end{array} \quad \begin{array}{l} \text{Stage 1(b)} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array} \end{array}$$

In this case, the back substitution only gives the trivial solution

$$x_1 = x_2 = x_3 = 0.$$

However, it is possible with some sets of homogeneous equations to obtain solutions other than the trivial solution, as the following example will demonstrate.

Example 2

Solve the simultaneous equations

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 0 \\ x_1 + 2x_2 + 3x_3 & = & 0 \\ 2x_1 + 3x_2 + 4x_3 & = & 0 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

The elimination gives us

$$\begin{array}{l} \text{Stage 1(a)} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array} \end{array} \quad \begin{array}{l} \text{Stage 1(b)} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array} \end{array}$$

The row of zeros in Stage 1(b) represents an 'equation' $0 = 0$. This 'loss' of an equation during the elimination process means that we can obtain solutions other than the trivial solution. This is because the equations we are left with,

$$x_1 + x_2 + x_3 = 0$$

 E_1

$$x_2 + 2x_3 = 0$$

 E_{2a}

have an infinite number of solutions, as we shall now show.

Let $x_3 = k$,

then from E_{2a} , $x_2 = -2k$

and from E_1 , $x_1 = k$.

So this set of equations has an infinite number of solutions of the form

$$x_1 = k, \quad x_2 = -2k, \quad x_3 = k,$$

where k is arbitrary.

In this last example, the reason why we obtained an infinite number of solutions was that the rows of $A|b$ were linearly dependent. In Example 1, the rows of $A|b$ were linearly independent, and as a result, we obtained only the trivial solution. We thus arrived at the following conclusion.

If a set of n homogeneous simultaneous equations in n unknowns are linearly dependent, then they have an infinite number of solutions. If not, the equations only have the trivial solution $x_1 = x_2 = \dots = x_n = 0$.

Exercise 1

Solve the following sets of simultaneous equations.

$2x_1 + 3x_2 - 5x_3 = 0$	E_1	$2x_1 + 3x_2 - 5x_3 = 0$	E_1
$3x_1 - 4x_2 + x_3 = 0$	E_2	$3x_1 - 4x_2 + x_3 = 0$	E_2
$5x_1 + x_2 + x_3 = 0$	E_3	$5x_1 - 18x_2 + 13x_3 = 0$	E_3

(Solution on p. 56)

It is worth noting in the examples and exercises above, that in those homogeneous problems in which there were an infinite number of solutions:

- (a) the solutions were in proportion to each other, and
- (b) the trivial solution was always one of these solutions (obtained by putting $k = 0$).

To conclude this subsection, we look at an interesting connection between a set of simultaneous equations, and the corresponding set of homogeneous equations.

Example 3

If we were given the set of equations

$$\left. \begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 3x_2 + 4x_3 &= 9 \end{aligned} \right\} \quad (1)$$

theoretically we could solve them as follows:

- (a) First solve the corresponding set of homogeneous equations

$$\left. \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 0 \end{aligned} \right\} \quad (2)$$

As we saw, in Example 2, these equations have the solution

$$x_1 = k, \quad x_2 = -2k, \quad x_3 = k,$$

where k is arbitrary.

(b) Now try to spot a *particular* solution of the original equations (1). In this case, for instance, we can see that $x_1 = x_2 = x_3 = 1$ is a solution to these equations.

From this information, we can now construct the general solution to the set of equations (2), as follows.

As for recurrence relations, and differential equations, the *general* solution to a set of equations can be formed by adding a *particular* solution of these to the solution of the corresponding homogeneous set. Hence in this case, we can write down the solution

$$x_1 = 1 + k, \quad x_2 = 1 - 2k, \quad x_3 = 1 + k.$$

Although this was a useful technique in the case of recurrence relations and differential equations, and actually works for linear simultaneous equations, it doesn't really help us much in the latter case. *Most* of the sets of equations we meet have unique solutions, and if we could spot the solutions to these, we would have no need for Gaussian elimination—and the problem would be done anyway!

In the unit on eigenvalues you will meet a situation in which homogeneous sets of equations arise naturally.

5.2 m equations in n unknowns ($m \neq n$)

This unit has been restricted to discussing a method of solving n equations in n unknowns. Real life is not always kind enough to provide us with this ideal situation! In fact, when faced with m equations in n unknowns, where $m \neq n$, the techniques used are rather different from the method described in this unit.

(a) More unknowns than equations

This is a situation which we already know provides us with an infinite number of solutions (as long as the equations are not inconsistent). For instance, if we met the equations

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 3 \\ x_2 + x_3 & = & 2 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \end{array}$$

this is a situation very similar to the one where we lost an equation after the elimination was complete. If we let $x_3 = k$, where k is arbitrary, we can write down the solution of the equations in terms of k :

from E_2 , $x_2 = 2 - k$, and from E_1 , $x_1 = 1$, giving a solution

$$x_1 = 1, \quad x_2 = 2 - k, \quad x_3 = k$$

where k is arbitrary.

Exercise 2

Solve the simultaneous equations

$$\begin{array}{rcl} x_1 + 2x_2 + 3x_3 & = & 4 \\ x_1 - x_2 + x_3 & = & 6 \end{array}$$

(Solution on p. 57)

In general, if there are more unknowns than equations, there will be an infinite number of solutions (except in the rare case of inconsistent equations, in which case there will be none).

(b) More equations than unknowns

If there are more equations than unknowns, then the equations are normally inconsistent. For example, the following system of three equations in two unknowns

$$\begin{array}{rcl} x_1 + x_2 & = & 2 \\ x_1 + 2x_2 & = & 3 \\ 2x_1 + 3x_2 & = & 6 \end{array} \quad \begin{array}{l} E_1 \\ E_2 \\ E_3 \end{array}$$

has no solution. The only values of x_1 and x_2 satisfying the first two equations are $x_1 = 1$, $x_2 = 1$; but these values do not satisfy the third equation, as you can verify for yourself.

However, if the third equation were changed to

$$2x_1 + 3x_2 = 5$$

$$E_3$$

then the values $x_1 = 1$, $x_2 = 1$ would satisfy all three equations. The reason why this is now possible is that the three equations are *linearly dependent*; in fact we have $E_3 = E_1 + E_2$, so that equation E_3 gives no new information that is not already in E_1 or E_2 . This possibility is, however, exceptional. During the Gaussian elimination process it would reveal itself by one of the equations reducing to $0 = 0$ at a certain stage.

Summary of Section 5

1. If all the right-hand sides of a set of simultaneous equations are zero (i.e. $\mathbf{b} = \mathbf{0}$), the equations are called **homogeneous**.
2. A set of n homogeneous equations in n unknowns will have:
 - (i) an infinite number of solutions if the rows of $\mathbf{A}|\mathbf{b}$ are linearly dependent;
 - (ii) only the all-zero solution (called **the trivial solution**) if the rows of $\mathbf{A}|\mathbf{b}$ are linearly independent.
3. A set of m equations in n unknowns will have:
 - (i) an infinite number of solutions if $m < n$ (unless the equations are inconsistent, in which case there are no solutions);
 - (ii) no solutions if $m > n$ (unless enough of the equations are linearly dependent for the system to reduce effectively to n equations or less).

6 End of unit test

Section A

These are multiple choice questions to which there is only one correct answer. Spend very little time on these. If you can't do them, or get the answers wrong, re-read the appropriate section(s) in the unit.

1. In one of the following, the rows of the matrix are linearly dependent. Which one?

(a) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 1 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 & 8 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 1 \end{bmatrix}$
2. Which of the following sets of simultaneous equations have no solution?

(a) $2x_1 + 3x_2 = 5$
 $x_1 + x_2 = 6$

(b) $2x_1 + 4x_2 = 0$
 $x_1 + x_2 = 0$

(c) $2x_1 + 4x_2 = 0$
 $x_1 + x_2 = 0$
 $x_1 + 2x_2 = 0$

(d) $2x_1 + 4x_2 = 8$
 $x_1 + 2x_2 = 4$

(e) $2x_1 + 3x_2 = 5$
 $x_1 + \frac{3}{2}x_2 = 6$
3. In which of the following sets of simultaneous equations would you expect unreliable solutions?

(a) $3.0x_1 + 2.4x_2 = 5.4$
 $2.2x_1 + 4.8x_2 = 7.6$

(b) $3.0x_1 + 2.4x_2 = 5.4$
 $5.9x_1 + 4.8x_2 = 10.8$

(c) $3.0x_1 + 2.4x_2 = 5.4$
 $9.8x_1 - 7.0x_2 = 16.2$
4. Choose the option which gives a true statement. Partial pivoting has the effect of:
 - (a) making sure that all multipliers are less than 1;
 - (b) eliminating ill-conditioning;
 - (c) reducing the chance of error build-up;
 - (d) eradicating the effects of rounding error.

Section B

These are longer questions.

5. Determine the nature of the solutions of the following sets of equations. In the case where the equations have a unique solution, find it. In the case where the equations have an infinite number of solutions:

(a) find the general solution;

(b) find which linear combination of the rows of $\mathbf{A}|\mathbf{b}$ gives a row of zeros.

$$\begin{array}{lll} \text{(i)} & x_1 - x_2 + x_3 = 7 & \text{(ii)} \quad x_1 - x_2 + x_3 = 7 \quad \text{(iii)} \quad x_1 - x_2 + x_3 = 7 \\ & 2x_1 + 3x_2 - x_3 = -6 & 2x_1 + 3x_2 - x_3 = -6 \quad 2x_1 + 3x_2 - x_3 = -6 \\ & 11x_1 + 9x_2 - x_3 = -2 & 3x_1 - 4x_2 - 2x_3 = -3 \quad 9x_1 + x_2 + 3x_3 = 23 \end{array}$$

6. Use Gaussian elimination with partial pivoting to solve the following equations.

$$3.1x_1 - 4.5x_2 + 7.1x_3 = 23.77$$

$$4.2x_1 + 1.2x_2 - 1.7x_3 = 29.95$$

$$1.5x_1 - 3.5x_2 + 3.1x_3 = 4.510$$

Do your work as accurately as your calculator will allow, but give your answer to two significant figures.

7. Find the solution(s) of the following sets of homogeneous simultaneous equations.

$$\begin{array}{lll} \text{(i)} & x_1 - x_2 + x_3 = 0 & \text{(ii)} \quad x_1 - x_2 + x_3 = 0 \quad \text{(iii)} \quad x_1 + 2x_2 + 3x_3 = 0 \\ & 2x_1 + 3x_2 - x_3 = 0 & 2x_1 + 3x_2 - x_3 = 0 \quad x_1 + 3x_2 - 5x_3 = 0 \\ & 11x_1 + 9x_2 - x_3 = 0 & 3x_1 - 4x_2 - 2x_3 = 0 \end{array}$$

[Hint: use the eliminations for Question 5(i) and (ii) to do the first two parts of this question.]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. Stage 1: 'eliminate' x_1 by taking $E_2 - 2E_1$, which gives

$$7x_2 = -14 \quad E_{2a}$$

Stage 2: from E_{2a} we find $x_2 = -2$, and substituting this into E_1 we get

$$x_1 + 4 = 5.$$

Hence $x_1 = 1$, giving the solution

$$x_1 = 1, \quad x_2 = -2.$$

Check

$$\text{LHS of } E_1: x_1 - 2x_2 = 1 - (2 \times (-2)) = 5 = \text{RHS.}$$

$$\text{LHS of } E_2: 2x_1 + 3x_2 = (2 \times 1) + (3 \times (-2)) = -4 = \text{RHS.}$$

2. Stage 1: using the rule stated, $E_2 - \frac{7}{5}E_1$ gives us

$$7x_1 - 3x_2 - \frac{7}{5}(5x_1 + 2x_2) = -1 - (\frac{7}{5} \times 20)$$

which, when simplified, gives

$$-\frac{29}{5}x_2 = -29 \quad E_{2a}$$

Stage 2: from E_{2a} we find $x_2 = 5$, and substituting this into E_1 gives us

$$5x_1 + 10 = 20.$$

Hence $x_1 = 2$, giving the solution

$$x_1 = 2, \quad x_2 = 5.$$

3. Stage 1(a): $E_2 - \frac{5}{3}E_1$ gives

$$-\frac{2}{3}x_2 + \frac{11}{3}x_3 = \frac{13}{3} \quad E_{2a}$$

and $E_3 - \frac{4}{3}E_1$ gives

$$-\frac{10}{3}x_2 - \frac{5}{3}x_3 = \frac{5}{3} \quad E_{3a}$$

Stage 1(b): $E_{3a} - (-\frac{10/3}{-2/3})E_{2a}$ gives

$$-\frac{60}{3}x_3 = -\frac{60}{3} \quad E_{3b}$$

Stage 2: from E_{3b} we find $x_3 = 1$, and substituting this into E_{2a} gives us

$$-\frac{2}{3}x_2 + \frac{11}{3} = \frac{13}{3}.$$

Hence $x_2 = -1$. Now from E_1 ,

$$3x_1 - 1 - 1 = 1.$$

Hence $x_1 = 1$, so the solution is

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1.$$

4.
$$\begin{bmatrix} 3 & -5 & 8 \\ 4 & 7 & 11 \end{bmatrix}$$

5. The matrix representing these equations is

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{matrix}$$

Stage 1:

$$\mathbf{R}_2 - 3\mathbf{R}_1 \begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -7 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \end{matrix}$$

Stage 2: writing out the equations represented by the new matrix, we get

$$\begin{aligned} x_1 + 2x_2 &= 4 & E_1 \\ -7x_2 &= -7 & E_{2a} \end{aligned}$$

From E_{2a} we find $x_2 = 1$, and substituting this into E_1 we get

$$x_1 + 2 = 4.$$

Hence $x_1 = 2$, giving the solution

$$x_1 = 2, \quad x_2 = 1.$$

6. The matrix representing these equations is

$$\begin{bmatrix} 3 & 1 & -1 \\ 5 & 1 & 2 \\ 4 & -2 & -3 \end{bmatrix} \begin{matrix} 1\mathbf{R}_1 \\ 6\mathbf{R}_2 \\ 3\mathbf{R}_3 \end{matrix}$$

Stage 1(a):

$$\begin{aligned} \mathbf{R}_2 - \frac{5}{3}\mathbf{R}_1 & \begin{bmatrix} 3 & 1 & -1 \\ 0 & -\frac{2}{3} & \frac{11}{3} \\ 4 & -2 & -3 \end{bmatrix} \begin{matrix} 1\mathbf{R}_1 \\ \frac{13}{3}\mathbf{R}_{2a} \\ \mathbf{R}_3 \end{matrix} \\ \mathbf{R}_3 - \frac{4}{3}\mathbf{R}_1 & \begin{bmatrix} 3 & 1 & -1 \\ 0 & -\frac{2}{3} & \frac{11}{3} \\ 0 & -\frac{10}{3} & -\frac{5}{3} \end{bmatrix} \begin{matrix} 1\mathbf{R}_1 \\ \frac{13}{3}\mathbf{R}_{2a} \\ \frac{5}{3}\mathbf{R}_{3a} \end{matrix} \end{aligned}$$

Stage 1(b):

$$\mathbf{R}_{3a} - (-\frac{10/3}{-2/3})\mathbf{R}_{2a} \begin{bmatrix} 3 & 1 & -1 \\ 0 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & -\frac{60}{3} \end{bmatrix} \begin{matrix} 1\mathbf{R}_1 \\ \frac{13}{3}\mathbf{R}_{2a} \\ -\frac{60}{3}\mathbf{R}_{3b} \end{matrix}$$

Stage 2: the rest of the solution is the same as in Exercise 3.

7.

(i) The matrix is

$$\begin{bmatrix} 1 & -2 & 5 \\ 4 & 1 & 2 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{matrix}$$

Stage 1:

$$\mathbf{R}_2 - 4\mathbf{R}_1 \begin{bmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{matrix}$$

Stage 2: the equations represented by the new matrix are

$$\begin{aligned} x_1 - 2x_2 &= 5 & E_1 \\ 9x_2 &= -18 & E_{2a} \end{aligned}$$

giving the solution

$$x_1 = 1, \quad x_2 = -2.$$

(ii) The matrix is

$$\begin{bmatrix} 2 & -3 & 16 \\ 3 & -4 & 22 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{matrix}$$

Stage 1:

$$\mathbf{R}_2 - \frac{3}{2}\mathbf{R}_1 \begin{bmatrix} 2 & -3 & 16 \\ 0 & \frac{1}{2} & -2 \end{bmatrix}$$

Stage 2: the equations represented by the new matrix are

$$\begin{aligned} 2x_1 - 3x_2 &= 16 \\ \frac{1}{2}x_2 &= -2 \end{aligned}$$

giving the solution

$$x_1 = 2, \quad x_2 = -4.$$

8. (i) The matrix is

$$\begin{bmatrix} 1 & -3 & 1 & 11 \\ 2 & 1 & 4 & 3 \\ 3 & -5 & -7 & 11 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{matrix}$$

Stage 1(a):

$$\begin{array}{l} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - 3\mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 11 \\ 0 & 7 & 2 & -19 \\ 0 & 4 & -10 & -22 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\mathbf{R}_{3a} - \frac{4}{7}\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 11 \\ 0 & 7 & 2 & -19 \\ 0 & 0 & -\frac{78}{7} & -\frac{78}{7} \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2: the equations represented by the new matrix are

$$\begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 11 \quad E_1 \\ 7x_2 + 2x_3 & = & -19 \quad E_{2a} \\ -\frac{78}{7}x_3 & = & -\frac{78}{7} \quad E_{3b} \end{array}$$

giving the solution

$$x_1 = 1, \quad x_2 = -3, \quad x_3 = 1.$$

(ii) The matrix is

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & 10 \\ 1 & 2 & 3 & 10 \\ 1 & 5 & -3 & 10 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a):

$$\begin{array}{l} \mathbf{R}_2 - \frac{1}{3}\mathbf{R}_1 \\ \mathbf{R}_3 - \frac{1}{3}\mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 3 & -2 & 5 & 10 \\ 0 & \frac{8}{3} & \frac{4}{3} & \frac{20}{3} \\ 0 & \frac{17}{3} & -\frac{14}{3} & \frac{20}{3} \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\mathbf{R}_{3a} - \frac{17}{8}\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 3 & -2 & 5 & 10 \\ 0 & \frac{8}{3} & \frac{4}{3} & \frac{20}{3} \\ 0 & 0 & -\frac{180}{24} & -\frac{180}{24} \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2: the equations represented by the new matrix are

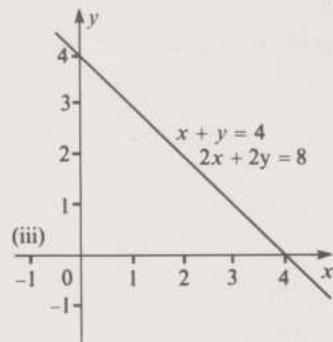
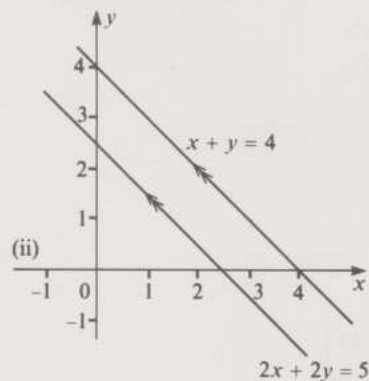
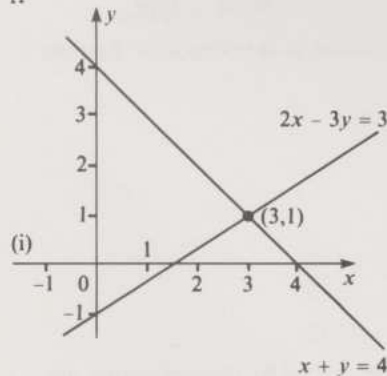
$$\begin{array}{rcl} 3x_1 - 2x_2 + 5x_3 & = & 10 \quad E_1 \\ \frac{8}{3}x_2 + \frac{4}{3}x_3 & = & \frac{20}{3} \quad E_{2a} \\ -\frac{180}{24}x_3 & = & -\frac{180}{24} \quad E_{3b} \end{array}$$

giving the solution

$$x_1 = 3, \quad x_2 = 2, \quad x_3 = 1.$$

Solutions to the exercises in Section 2

1.



2.

(i) Stage 1(a):

$$\begin{array}{l} \mathbf{R}_2 - \frac{3}{2}\mathbf{R}_1 \\ \mathbf{R}_3 - \frac{1}{2}\mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 8 \\ 0 & \frac{1}{2} & -\frac{7}{2} & -6 \\ 0 & \frac{3}{2} & -\frac{21}{2} & -2 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\mathbf{R}_{3a} - 3\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 8 \\ 0 & \frac{1}{2} & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 16 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

From $\mathbf{R}_{3b}$ we get $0 = 16$, so the given equations are inconsistent and have no solution.

(ii) Stage 1(a):

$$\begin{array}{l} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - \mathbf{R}_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 8 & -9 & -17 \\ 0 & 16 & -18 & -34 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\mathbf{R}_{3a} - 2\mathbf{R}_{2a} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 8 & -9 & -17 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2: the equations represented by the new matrix are

$$\begin{array}{rcl} x_1 - 2x_2 + 4x_3 & = & 7 \quad E_1 \\ 8x_2 - 9x_3 & = & -17 \quad E_{2a} \end{array}$$

Let $x_3 = k$. Then from E_{2a} ,

$$x_2 = \frac{-17 + 9k}{8}$$

and from E_1 ,

$$\begin{aligned} x_1 &= 7 + 2 \left(\frac{-17 + 9k}{8} \right) - 4k \\ &= \frac{11 - 7k}{4}. \end{aligned}$$

Thus there are an infinite number of solutions, the general solution being

$$x_1 = \frac{11 - 7k}{4}, \quad x_2 = \frac{-17 + 9k}{8}, \quad x_3 = k$$

where k is arbitrary.

(iii) Stage 1(a):

$$\begin{array}{ccc|c} 3 & -2 & 3 & 27 \\ R_2 - \frac{2}{3}R_1 & 0 & \frac{13}{3} & -7 \\ R_3 - \frac{4}{3}R_1 & 0 & \frac{14}{3} & -3 \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{ccc|c} 3 & -2 & 3 & 27 \\ 0 & \frac{13}{3} & -7 & \frac{8}{3} \\ R_{3a} - \frac{14}{13}R_{2a} & 0 & \frac{59}{13} & -\frac{59}{13} \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

Stage 2: the equations represented by the new matrix are

$$\begin{aligned} 3x_1 - 2x_2 + 3x_3 &= 27 \\ \frac{13}{3}x_2 - 7x_3 &= \frac{8}{3} \\ \frac{59}{13}x_3 &= -\frac{59}{13} \end{aligned}$$

giving the unique solution

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = -1.$$

3. In Exercise 2(ii) we found

$$R_{3b} = R_{3a} - 2R_{2a} \quad (1)$$

where

$$R_{2a} = R_2 - 2R_1 \text{ and } R_{3a} = R_3 - R_1.$$

Substituting these expressions into (1) gives

$$R_{3b} = 3R_1 - 2R_2 + R_3.$$

Hence,

$$3R_1 - 2R_2 + R_3 = 0.$$

4. Stage 1(a):

$$\begin{array}{ccc|c} 1 & -2 & 5 & 7 \\ R_2 - R_1 & 0 & 5 & -9 \\ R_3 - R_1 & 0 & 20 & -36 \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{ccc|c} 1 & -2 & 5 & 7 \\ 0 & 5 & -9 & 13 \\ R_{3a} - 4R_{2a} & 0 & 0 & -19 \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

From R_{3b} we get $0 = -19$, so the given equations are inconsistent and have no solution.

5. Stage 1(a):

$$\begin{array}{ccc|c} 1 & -2 & 5 & 6 \\ R_2 - R_1 & 0 & 5 & -9 \\ R_3 - 2R_1 & 0 & 11 & -22 \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{ccc|c} 1 & -2 & 5 & 6 \\ 0 & 5 & -9 & 1 \\ R_{3a} - \frac{11}{5}R_{2a} & 0 & 0 & -\frac{11}{5} \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

Back substitution gives the unique solution

$$x_1 = 5, \quad x_2 = 2, \quad x_3 = 1.$$

6. Stage 1(a):

$$\begin{array}{ccc|c} 5 & -1 & -1 & 2 \\ R_2 - \frac{1}{5}R_1 & 0 & -\frac{19}{5} & \frac{6}{5} \\ R_3 - \frac{6}{5}R_2 & 0 & \frac{76}{5} & -\frac{272}{5} \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{ccc|c} 5 & -1 & -1 & 2 \\ 0 & -\frac{19}{5} & \frac{6}{5} & \frac{68}{5} \\ R_{3a} + 4R_{2a} & 0 & 0 & 0 \end{array} \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} \end{array}$$

(i) As $R_{3b} = 0$, there will be an infinite number of solutions, given by the equations

$$\begin{aligned} 5x_1 - x_2 - x_3 &= 2 & E_1 \\ -\frac{19}{5}x_2 + \frac{6}{5}x_3 &= \frac{68}{5} & E_{2a} \end{aligned}$$

Let $x_3 = k$. Then from E_{2a} ,

$$x_2 = \frac{6k - 68}{19}$$

and from E_1 ,

$$\begin{aligned} x_1 &= \frac{1}{5} \left(2 + \frac{6k - 68}{19} + k \right) \\ &= \frac{5k - 6}{19}. \end{aligned}$$

Thus the general solution is

$$x_1 = \frac{5k - 6}{19}, \quad x_2 = \frac{6k - 68}{19}, \quad x_3 = k$$

where k is arbitrary.

(ii) From Stage 1(b),

$$R_{3a} + 4R_{2a} = 0 \quad (1)$$

where from Stage 1(a),

$$R_{2a} = R_2 - \frac{1}{5}R_1 \text{ and } R_{3a} = R_3 - \frac{6}{5}R_1.$$

Substituting these expressions into (1) gives

$$-2R_1 + 4R_2 + R_3 = 0.$$

Solutions to the exercises in Section 3

1. No solution given.

2. Using three figure arithmetic,

$$\begin{aligned} (0.2745 \times 73.841) &= 6.1879 \\ &\approx (0.275 \times 73.8) = 6.19 \\ &\approx 20.3 - 6.19 \\ &\approx 14.1. \end{aligned}$$

This compares well with the exact answer, 14.081455.

3. The matrix representing the equation is

$$\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 2 & -2 & 3 & 2 & 5 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & -1 & 4 & 1 & 4 \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

Stage 1(a):

$$\begin{array}{c} 2 \\ 0 \\ 0 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & -1 & 4 & 1 & 4 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \\ \mathbf{R}_{4a} \end{array}$$

As the second pivot is zero, change $\mathbf{R}_{2a}$ with $\mathbf{R}_{3a}$ (or with $\mathbf{R}_{4a}$):

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & -1 & 4 & 1 & 4 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \leftarrow \\ \mathbf{R}_{3a} \leftarrow \\ \mathbf{R}_{4a} \end{array}$$

Now we can proceed with the next stage of the elimination.

Stage 1(b):

$$\begin{array}{c} 0 \\ -1 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 2 & -2 & 4 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \\ \mathbf{R}_{4b} \end{array}$$

Stage 1(c):

$$\begin{array}{c} 2 \\ 0 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & -10 & 10 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \\ \mathbf{R}_{4c} \end{array}$$

Stage 2: back substitution gives the solution

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1, \quad x_4 = -1.$$

4. The matrix representing the equations is

$$\left[\begin{array}{ccc|c} 1.62 & 1.10 & 0.65 & 3.37 \\ 6.18 & 4.20 & -3.04 & 7.34 \\ 4.65 & -3.05 & 2.10 & 3.70 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

(i) Using four-figure arithmetic, and no partial pivoting:

Stage 1(a):

$$\begin{array}{c} 3.815 \\ 2.870 \end{array} \left[\begin{array}{ccc|c} 1.62 & 1.1 & 0.65 & 3.37 \\ 0 & 0.003 & -5.52 & -5.52 \\ 0 & -6.207 & 0.234 & -5.972 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{c} -2069 \end{array} \left[\begin{array}{ccc|c} 1.620 & 1.1 & 0.65 & 3.37 \\ 0 & 0.003 & -5.52 & -5.52 \\ 0 & 0 & -11420 & -11430 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2: from $\mathbf{R}_{3b}$ we find

$$x_3 = 1.001 \text{ (to four significant figures).}$$

Substituting this into the equation described by $\mathbf{R}_{2a}$ gives us $x_2 = 2.000$ (to four significant figures).

Finally from $\mathbf{R}_1$ we get

$$x_1 = 0.3206 \text{ (to four significant figures).}$$

Thus the solution we obtain by this method,

$$x_1 = 0.3206, \quad x_2 = 2.000, \quad x_3 = 1.001,$$

is nothing like the correct solution.

(ii) Again using four-figure arithmetic, but this time using partial pivoting, first we interchange $\mathbf{R}_1$ and $\mathbf{R}_2$:

$$\left[\begin{array}{ccc|c} 6.18 & 4.20 & -3.04 & 7.34 \\ 1.62 & 1.10 & 0.65 & 3.37 \\ 4.65 & -3.05 & 2.10 & 3.70 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \leftarrow \\ \mathbf{R}_2 \leftarrow \\ \mathbf{R}_3 \end{array}$$

Stage 1(a):

$$\begin{array}{c} 0.2621 \\ 0.7524 \end{array} \left[\begin{array}{ccc|c} 6.18 & 4.20 & -3.04 & 7.34 \\ 0 & -0.001 & 1.447 & 1.446 \\ 0 & -6.21 & 4.387 & -1.823 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Now $\mathbf{R}_{2a}$ and $\mathbf{R}_{3a}$ must be interchanged:

$$\left[\begin{array}{ccc|c} 6.18 & 4.20 & -3.04 & 7.34 \\ 0 & -6.21 & 4.387 & -1.823 \\ 0 & -0.001 & 1.447 & 1.446 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \leftarrow \\ \mathbf{R}_{3a} \leftarrow \end{array}$$

Stage 1(b):

$$\begin{array}{c} 0.0002 \end{array} \left[\begin{array}{ccc|c} 6.18 & 4.20 & -3.04 & 7.34 \\ 0 & -6.21 & 4.387 & -1.823 \\ 0 & 0 & 1.446 & 1.446 \end{array} \right] \begin{array}{l} \mathbf{R} \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2: proceeding exactly as in part (i), this time we obtain

$$x_1 = 1.000, \quad x_2 = 1.000, \quad x_3 = 1.000,$$

which is actually correct (to four significant figures, of course!)

Solutions to the exercises in Section 5

1.

(i) Stage 1(a):

$$\begin{array}{c} \frac{3}{2} \\ \frac{5}{2} \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -5 & 0 \\ 0 & -\frac{17}{2} & \frac{17}{2} & 0 \\ 0 & -\frac{13}{2} & \frac{27}{2} & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{c} \frac{13}{17} \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -5 & 0 \\ 0 & -\frac{17}{2} & \frac{17}{2} & 0 \\ 0 & 0 & \frac{14}{2} & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

This gives only the trivial solution

$$x_1 = x_2 = x_3 = 0.$$

(ii) Stage 1(a):

$$\begin{array}{c} \frac{3}{2} \\ \frac{5}{2} \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -5 & 0 \\ 0 & -\frac{17}{2} & \frac{17}{2} & 0 \\ 0 & -\frac{51}{2} & \frac{51}{2} & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b):

$$\begin{array}{c} 3 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & -5 & 0 \\ 0 & -\frac{17}{2} & \frac{17}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

As the rows are linearly dependent, we get an infinite number of solutions

$$x_1 = x_2 = x_3 = k$$

where k is arbitrary.

2. The matrix is

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 1 & -1 & 1 & | & 6 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{matrix}$$

Do one stage of the elimination:

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & -3 & -2 & | & 2 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \end{matrix}$$

The equations represented by this matrix are

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 & E_1 \\ -3x_2 - 2x_3 &= 2 & E_{2a} \end{aligned}$$

Let $x_3 = k$. Then from E_{2a} ,

$$x_2 = \frac{-(2k + 2)}{3}$$

and from E_1 ,

$$\begin{aligned} x_1 &= 4 - 2 \left(\frac{-(2k + 2)}{3} \right) - 3k \\ &= \frac{-5k + 16}{3}. \end{aligned}$$

Thus there are an infinite number of solutions, the general solution being

$$x_1 = \frac{-5k + 16}{3}, \quad x_2 = \frac{-(2k + 2)}{3}, \quad x_3 = k$$

where k is arbitrary.

Solutions to the end of unit test

Section A

- The answer is (c), because $\mathbf{R}_1 = 2\mathbf{R}_2$ in this matrix.
- The answer is (e). Multiplying E_2 of this set by 2 gives an equation which is obviously inconsistent with E_1 .
- The answer is (b). $\mathbf{R}_2 \approx 2\mathbf{R}_1$ in this matrix, so a small change in the initial data is likely to produce a large change in the solution. Another way of asking this question would have been: which set of equations is ill-conditioned?
- The answer is (c). Option (a) is wrong because it would allow a multiplier to be -500 , which would be most undesirable! Option (b) is wrong because no technique of solution can eliminate ill-conditioning. Option (d) is wrong because the effects of rounding error cannot be totally eradicated by partial pivoting—only reduced.

Section B

5.

(i) Stage 1(a):

$$\begin{bmatrix} 1 & -1 & 1 & | & 7 \\ 2 & 0 & 5 & -3 & | & -20 \\ 11 & 0 & 20 & -12 & | & -79 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{matrix}$$

Stage 1(b):

$$\begin{bmatrix} 1 & -1 & 1 & | & 7 \\ 0 & 5 & -3 & | & -20 \\ 4 & 0 & 0 & | & 1 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{matrix}$$

From $\mathbf{R}_{3b}$, we see that these equations have no solution.

(ii) Stage 1(a)

$$\begin{bmatrix} 1 & -1 & 1 & | & 7 \\ 2 & 0 & 5 & -3 & | & -20 \\ 3 & 0 & -1 & -5 & | & -24 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{matrix}$$

Stage 1(b)

$$\begin{bmatrix} 1 & -1 & 1 & | & 7 \\ 0 & 5 & -3 & | & -20 \\ -\frac{1}{3} & 0 & -\frac{28}{3} & | & -28 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{matrix}$$

Back substitution gives the unique solution

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 5.$$

(iii) Stage 1(a):

$$\begin{bmatrix} 1 & -1 & 1 & | & 7 \\ 2 & 0 & 5 & -3 & | & -20 \\ 9 & 0 & 10 & -6 & | & -40 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{matrix}$$

Stage 1(b):

$$\begin{bmatrix} 1 & -1 & 1 & | & 7 \\ 0 & 5 & -3 & | & -20 \\ 2 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{matrix}$$

These equations have an infinite number of solutions.

(a) Letting $x_3 = k$, back substitution gives the general solution

$$x_1 = \frac{15 - 2k}{5}, \quad x_2 = \frac{3k - 20}{5}, \quad x_3 = k$$

where k is arbitrary.

(b) From Stage 1(b),

$$\mathbf{R}_{3a} - 2\mathbf{R}_{2a} = \mathbf{0} \tag{1}$$

where from Stage 1(a),

$$\mathbf{R}_{2a} = \mathbf{R}_2 - 2\mathbf{R}_1 \text{ and } \mathbf{R}_{3a} = \mathbf{R}_3 - 9\mathbf{R}_1.$$

Substituting these expressions into (1) gives

$$-5\mathbf{R}_1 - 2\mathbf{R}_2 + \mathbf{R}_3 = \mathbf{0}.$$

6. As calculators differ in the degree of precision to which they work, the following numbers are just supposed to act as guide lines.

The largest entry in the first column is in row 2, so we exchange rows 1 and 2 to obtain the matrix

$$\begin{bmatrix} 4.2 & 1.2 & -1.7 & | & 29.95 \\ 3.1 & -4.5 & 7.1 & | & 23.77 \\ 1.5 & -3.5 & 3.1 & | & 4.510 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \uparrow \\ \mathbf{R}_2 \leftarrow \\ \mathbf{R}_3 \end{matrix}$$

Stage 1(a):

$$\begin{bmatrix} 4.2 & 1.2 & -1.7 & | & 29.95 \\ 0.738 & 0 & -5.39 & | & 1.66 \\ 0.357 & 0 & -3.93 & | & -6.19 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{matrix}$$

Stage 1(b):

$$\begin{bmatrix} 4.2 & 1.2 & -1.7 & | & 29.95 \\ 0 & -5.39 & 8.35 & | & 1.66 \\ 0.729 & 0 & -2.39 & | & -7.40 \end{bmatrix} \begin{matrix} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{matrix}$$

Back substitution now gives, to two significant figures, the solution

$$x_1 = 7.1, \quad x_2 = 4.5, \quad x_3 = 3.1.$$

7. (i) From Question 5(i), the matrix obtained after the elimination is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Letting $x_3 = k$, back substitution gives the general solution

$$x_1 = -\frac{2}{5}k, \quad x_2 = \frac{3}{5}k, \quad x_3 = k$$

where k is arbitrary.

(ii) From Question 5(ii), the matrix obtained after the elimination is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & -\frac{28}{5} & 0 \end{array} \right]$$

From this, we know that the equations have only the trivial solution

$$x_1 = x_2 = x_3 = 0.$$

(iii) As we have only two equations in three unknowns, if a solution exists it will not be unique. Furthermore, as the equations are homogeneous we know that a solution does exist, namely the trivial solution.

To find the general solution, do one stage of elimination. This produces the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -8 & 0 \end{array} \right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \end{array}$$

Letting $x_3 = k$, back substitution gives the general solution

$$x_1 = -19k, \quad x_2 = 8k, \quad x_3 = k$$

where k is arbitrary.

